REPORT 1099

AIR FORCES AND MOMENTS ON TRIANGULAR AND RELATED WINGS WITH SUBSONIC LEADING EDGES OSCILLATING IN SUPERSONIC POTENTIAL FLOW [1]

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SUMMARY

This analysis treats the air forces and moments in supersonic potential flow on oscillating triangular wings and a series of sweptback and arrow wings with subsonic leading edges and supersonic trailing edges. For the wings undergoing sinusoidal torsional oscillations simultaneously with vertical translations, the linearized velocity potential is derived in the form of a power series in terms of a frequency parameter. This method can be useful for treatment of similar problems for other plan forms and for wings undergoing other sinusoidal motions. For triangular wings, as many terms of such a series expansion as may be desired can be determined; however, the terms after the first few become very cumbersome.

Closed expressions that include the reduced frequency to the fifth power, an order which is sufficient for a large class of practical applications, are given for the velocity potential and for the components of chordwise section force and moment coefficients.

These wings are found to exhibit the possibility of undamped torsional oscillations for certain ranges of Mach number and locations of the axis of rotation. The ranges of these parameters are delineated for triangular wings.

INTRODUCTION

This report is concerned with the derivation of expressions for the velocity potential and associated forces and moments for oscillating triangular wings in supersonic flow. The boundary-value problem for the linearized velocity potential for an apex-forward triangular wing oscillating in a supersonic main stream may be classified, according to reference 1. as "purely supersonic" if the leading edges of the triangle are outside the Mach cone emanating from the apex of the triangle or "mixed supersonic" if the leading edges are inside this Mach cone.

In the purely supersonic case the principle of independence holds; that is, the flow on the upper surface of the wing is independent of the flow on the lower surface and vice versa. Garrick and Rubinow (reference 1) have shown that the boundary-value problem for the velocity potential in the purely supersonic case can be satisfied by simple distributions of sources with local strength proportional to the local prescribed normal velocity of the wing. Nelson (reference 2) has treated the oscillating triangular wing for this case to the third power of the frequency.

In the mixed supersonic case the principle of independence does not hold. Boundary-value problems for lift-producing wings in this case can be satisfied by distributions of doublets; the relation between doublet strength and normal velocity of the wing is, however, in general, not simple. The determination of this relation requires the solution of an integral equation that employs the potential of a time-dependent unit doublet as kernel and limits of integration that depend on Mach number and wing plan form.

For treatment of problems that involve boundary conditions that are independent of time—such as constant angle of attack, constant rate of pitching, and so forth—the doublet potential, employed as kernel of the integral equation, is considered independent of time and in these cases the integral equations for triangular wings can be solved by a straightforward process.

For treatment of problems of oscillating wings, however, it is necessary to employ, as the kernel of the integral equations, a doublet potential that varies harmonically with time and in this case the solution of the integral equation, generally, becomes very cumbersome. If the doublet potential or kernel is expanded in terms of the frequency of oscillation, however, use can be made of knowledge of solutions of integral equations for problems that are independent of time to obtain an expanded form of solution for a wing undergoing harmonic oscillations. Such a procedure was demonstrated in treatments of rectangular wings in references 3 and 4.

The purpose of the present report is to make use of the expanded form of the velocity potential to obtain the forces and moments, based on the first few terms of this potential, for a rigid triangular wing performing vertical and pitching sinusoidal oscillations in mixed supersonic flow. Although as many terms of the expanded potential as may be desired can be obtained after the first few terms, the process becomes very cumbersome. The flow normal to the leading edge is subsonic but the flow normal to the trailing edge is considered to be supersonic. This latter consideration implies that the potential derived for triangular plan forms may be used to calculate the aerodynamic forces and moments for other plan forms that may be formed with the triangular wing by cutting the trailing edges so that they lie ahead of the Mach cones emanating from their foremost points.

Other approaches to the solution of the problem of oscillating triangular wings have been given by Robinson (reference 5), Haskind and Falkovich (reference 6), and by Stewartson

(reference 7). In both references 5 and 6 formal solutions to the problem were obtained in terms of special systems of curvilinear coordinates. Robinson's solution was given in terms of a double summation of trilinear combinations of Bessel functions of the first kind with Lamé functions of the first and second kinds. Similarly, the solution of Haskind and Falkovich was given in terms of summations of Bessel functions of the first kind combined with elliptic integrals of the first and second kinds. In both references 5 and 6 the potentials were not reduced to useful forms for calculating forces and moments.

In reference 7 Stewartson makes an interesting though specialized use of the Laplace transformation to develop a method whereby terms of the velocity potential for triangular plan forms, expanded as herein, can be obtained. Stewartson gives formulas that, except for errors presumably in printing, can be used to develop the potential to the second power of the frequency, but he omits many details in his derivation.

SYMBOLS

ϕ	disturbance-velocity potential
x,y,z	rectangular coordinates attached to wing moving in negative x-direction
ξ,η	rectangular coordinates used to represent space location of doublets in xy-plane
Z_m	function defining mean ordinates of any chordwise section of wing such as $y=y_1$ as shown in figure 1
$w(x,y_1,t)$	vertical velocity at surface of wing along chordwise section at $y=y_1$
x_0	abscissa of axis of rotation of wing as shown in figure 1
t	time
h	vertical displacement of axis of rotation
h_0	amplitude of vertical displacement of axis of rotation, positive downward
α	angle of attack
α_0	amplitude of angular displacement about axis of rotation, positive leading edge up
$\dot{h},\dot{\alpha}$	time derivatives of h and α, respectively
V	velocity of main stream
c	velocity of sound
M	free-stream Mach number (V/c)
$\beta=\sqrt{M^2-1}$	
ω	frequency of oscillations
$\bar{\omega}=\dfrac{M^2\omega}{V\beta^2}$	
k	reduced frequency $(b\omega/V)$
ϵ	half apex angle
$C=\tan\epsilon$	
$a_{n m}$	represents functions of $\bar{\omega}$, x, and M
D_n,F_n	functions used to denote doublet distribution functions
$W^2_{n,m},\overline{W}^2_{n,m},\overline{\overline{W}}^2_{n,m}$	constants associated with D_n depending on βC

A_j,P_j,Q_j,R_j	constants depending on βC
σ_j	constants depending on βC and M
$2b$	root chord of wing
s	semispan of wing
σ	slope of ray passing through vertex of wing
q,N,θ,τ	dummy variables
ρ	density
Δp	local pressure difference
P	section force (total force at any spanwise station)
L_1,L_2,L_3,L_4	components of section force coefficients
M_α	section moment (total moment about $x=x_0$ at any spanwise station)
M_1,M_2,M_3,M_4	components of section moment coefficients
$\overline{M}_4$	total component of damping-moment coefficient
$\rho_0=\beta C$	
F',E'	complete elliptic integrals of the first and second kinds, respectively, with moduli $\sqrt{1-\rho_0^2}$

ANALYSIS

BOUNDARY-VALUE PROBLEM FOR THE VELOCITY POTENTIAL

Referred to a rectangular coordinate system moving forward at a uniform supersonic speed in the negative x-direction (see fig. 1), the differential equation for the propagation of small disturbances that must be satisfied by the velocity potential is

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} \tag{1}$$

The main governing boundary condition to be satisfied by the velocity potential is that the flow be tangent to the surface of the wing or

$$\left(\frac{\partial\phi}{\partial z}\right)_{z\to0}=w(x,y,t)=V\frac{\partial Z_m}{\partial x}+\frac{\partial Z_m}{\partial t} \tag{2}$$

where Z_m is the vertical displacement of any point of the wing. For the particular case of a wing independently performing small sinusoidal pitching oscillations of amplitude α_0 about some spanwise axis x_0 and small sinusoidal vertical translations of amplitude h_0, the quantity Z_m in equation (2) is

$$Z_m=e^{i\omega t}[\alpha_0(x-x_0)+h_0]=\alpha(x-x_0)+h \tag{3}$$

(See fig. 1 (b) for sketch showing instantaneous displacement of section $y=y_1$.) For convenience, the frequency of oscillation of both pitching and translatory motion is denoted by ω. Considering these motions to occur at separate frequencies would add no difficulties to the derivation.

Substituting the expression for Z_m (equation (3)) into equation (2) gives

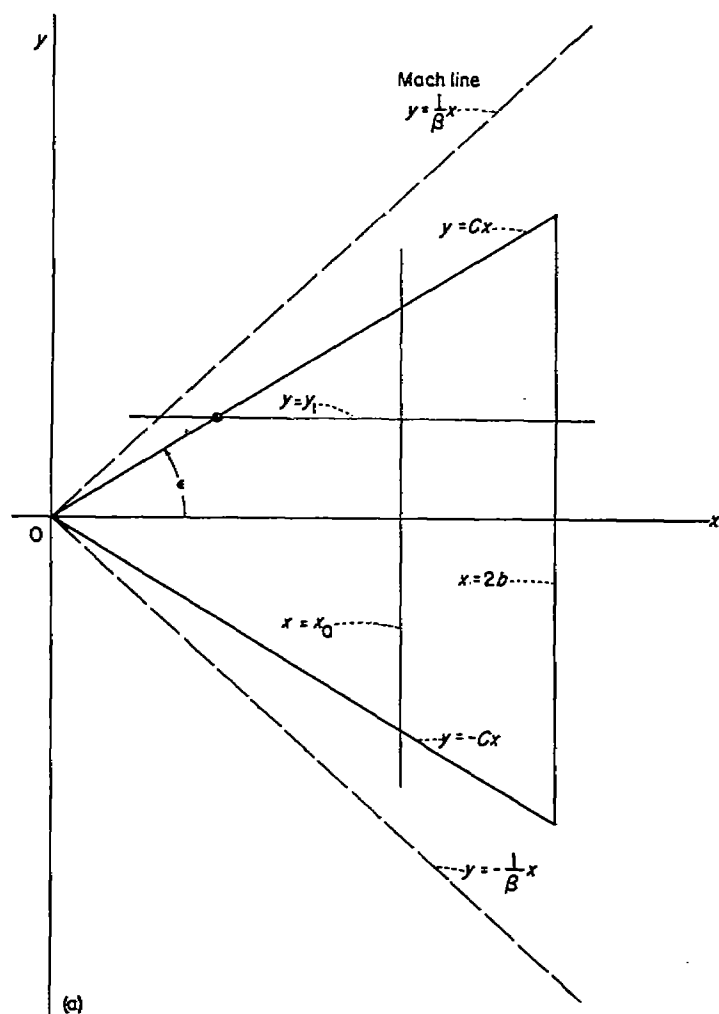$$w(x,y,t)=V\alpha+\dot{\alpha}(x-x_0)+\dot{h} \tag{4}$$

Equation (4) implies that the velocity potential may be

expressed as the sum of separate effects due to position and motion of the wing associated with individual terms of this equation, namely

$$\phi = \phi_\alpha + \phi_{\dot\alpha} + \phi_h \tag{5}$$

DERIVATION OF ϕ

In order to obtain the analytical expression for the potential ϕ, it is necessary to derive only one of the subsidiary potentials appearing in equation (5), say ϕ_α. The other subsidiary potentials $\phi_{\dot\alpha}$ and ϕ_h can then be obtained from the derived expression for ϕ_α by simple comparison.

In order to satisfy the boundary-value problem for ϕ_α, a convenient procedure is to start with the expanded form of the potential of a uniform distribution of doublets. Then, for a given power of the frequency of oscillation this potential, as will be shown in the following analysis, can be modified so that, when integration is made over the appropriate region, the results satisfy the differential equation (1) to the given power of the frequency and satisfy the condition of tangential flow exactly. The type of doublet required is that with its axis normal to the plane of the wing. The potential of such a doublet may be obtained from the potential of a source, located in the plane of the wing, by partial differentiation with respect to the direction normal to the plane of the wing. Similarly, the potential of a distribution of the required type of doublets can be obtained from a distribution of sources located in the plane of the wing.

The potential at (x,y,z) due to sources located at points $(\xi,\eta,0)$ in region r (illustrated in fig. 2) of the xy-plane which satisfies the differential equation (1) may be written as

$$\phi_s = \frac{V\alpha}{\pi} \int_r \int \frac{e^{-i\bar\omega(x-\xi)} \cos\left(\frac{\bar\omega}{M}R\right)}{R}\, d\xi\, d\eta \tag{6}$$

where

$$\bar\omega = \frac{M\omega}{c\beta^2} = \frac{M^2\omega}{V\beta^2}$$

and

$$R = \sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2 z^2}$$

(a) Plan form (xy-plane).
(b) Section $y=y_1$ (xz-plane).

FIGURE 1.—Sketch illustrating coordinate system and the two degrees of freedom α and h.
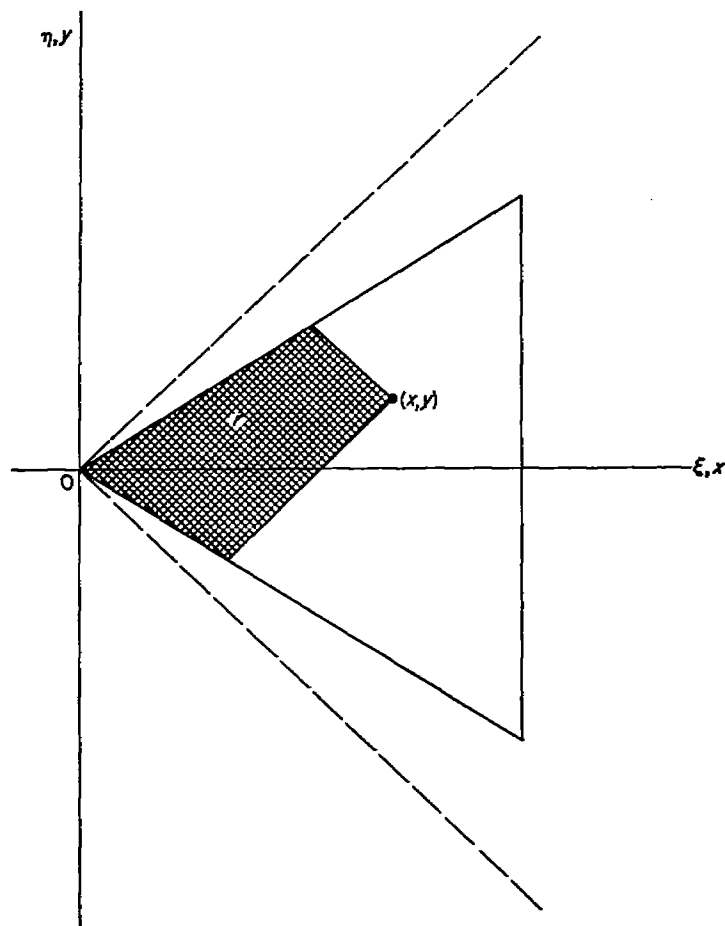

FIGURE 2.—Sketch illustrating region of integration for the velocity potential.

Expanding the integrand of equation (6) into a power series in $\bar{\omega}$, collecting terms with respect to ξ, and differentiating the resulting integral with respect to z gives the expanded form of the potential of a uniform distribution of doublets, namely

$$\phi_D = \frac{V\alpha}{\pi} \frac{\partial}{\partial z} \int_r \int \left[a_{01}\frac{1}{R} + a_{02}R + \ldots + a_{0m}R^{2m-3} + \ldots + \xi\left(a_{11}\frac{1}{R} + a_{12}R + \ldots + a_{1m}R^{2m-3} + \ldots \right) + \ldots + \right.$$

$$\left. \xi^n\left(a_{n1}\frac{1}{R} + a_{n2}R + \ldots + a_{nm}R^{2m-3} + \ldots \right) \right] d\xi\, d\eta \tag{7}$$

where

$$a_{n1} = \frac{(i\bar{\omega})^n}{n!} e^{-i\bar{\omega}x} \tag{8}$$

and

$$a_{nm} = \frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M}\right)^{2m-2} a_{n1} \tag{9}$$

For convenience in the succeeding discussion and analysis equation (7) may be written in the following form:

$$\phi_D = \frac{V\alpha}{\pi} \frac{\partial}{\partial z} \sum_{n=0}^{\infty}\left[a_{n1}\int_r\int \frac{\xi^n}{R}\, d\xi\, d\eta + \sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M}\right)^{2m-2} a_{n1}\int_r\int \xi^n R^{2m-3}\, d\xi\, d\eta \right] \tag{10}$$

An interesting and significant property of equation (10) is that the coefficient of each power of ξ satisfies the differential equation (1) and has the form of a source potential with strength proportional to $\left(\frac{i\bar{\omega}}{n!}\right)^n e^{-i\bar{\omega}\xi}$. This property may be shown by writing the coefficient of ξ^n as follows:

$$\frac{a_{n1}}{R}\left[1 + \sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M}R\right)^{2m-2}\right] = \frac{a_{n1}}{R}\cos\left(\frac{\bar{\omega}}{M}R\right)$$

$$= \left(\frac{i\bar{\omega}}{n!}\right)^n e^{-i\bar{\omega}\xi}\left[\frac{e^{-i\bar{\omega}(x-\xi)}\cos\left(\frac{\bar{\omega}}{M}R\right)}{R}\right] \tag{11}$$

A more general solution to equation (1) may thus be obtained by introducing properly chosen weight or distribution functions (denoted by $\xi D_n(\xi,\eta)$) into the coefficients of ξ^n in equation (10). Let this solution be denoted by ϕ_1; then it can be written as

$$\phi_1 = \frac{V\alpha}{\pi} \frac{\partial}{\partial z} \sum_{n=0}^{\infty}\left[a_{n1}\int_r\int D_n(\xi,\eta)\frac{\xi^{n+1}}{R}\, d\xi\, d\eta + \sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M}\right)^{2m-2} a_{n1}\int_r\int D_n(\xi,\eta)\xi^{n+1}R^{2m-3}d\xi\, d\eta \right] \tag{12}$$

Examination of equation (12) shows that, at the surface $z=0$, the potential ϕ_1 is determined by the first integral expression but that both integral expressions may give rise to normal velocity. In succeeding steps in this analysis it is shown that the distribution functions $D_n(\xi,\eta)$ in equation (12) can be determined so that the first integral expression taken alone will exactly satisfy the boundary condition of tangential flow for ϕ_α; that is $\left(\frac{\partial\phi_\alpha}{\partial z}\right)_{z=0} = V\alpha$. Also, any additional normal velocity that arises from the second integral expression can be canceled, to the required order, by consideration of additional doublet solutions to equation (1). The problem of satisfying the boundary-value problem for the velocity potential ϕ_α may thus be reduced to that of determining the appropriate distribution functions and additional solutions to equation (1).

In order to show that the first integral expression in equation (12) can be made to satisfy the boundary condition for ϕ_α, the coefficient $V\alpha a_{n1}$ appearing in this equation is first considered. If the analytical expression for the coefficient a_{n1} (equation (8)) is multiplied by x^n and summed with respect to n, the result is identically $V\alpha$. This result may be shown as follows:

$$V\alpha \sum_{n=0}^{\infty} x^n a_{n1} = V\alpha \sum_{n=0}^{\infty}\frac{(i\bar{\omega}x)^n}{n!} e^{-i\bar{\omega}x} = V\alpha e^{i\bar{\omega}x} e^{-i\bar{\omega}x} \equiv V\alpha \tag{13}$$

It is significant that this identity holds if only terms in $\bar{\omega}$ to any given power are considered. For example, retaining only terms including $\bar{\omega}$ to the fifth power gives

$$V\alpha\left[\left(1 - i\bar{\omega}x - \frac{\bar{\omega}^2x^2}{2} + \frac{i\bar{\omega}^3x^3}{6} + \frac{\bar{\omega}^4x^4}{24} - \frac{i\bar{\omega}^5x^5}{120}\right) + x\left(i\bar{\omega} + \bar{\omega}^2x - \frac{i\bar{\omega}^3x^2}{2} - \frac{\bar{\omega}^4x^3}{6} + \frac{i\bar{\omega}^5x^4}{24}\right) + x^2\left(-\frac{\bar{\omega}^2}{2} + \frac{i\bar{\omega}^3x}{2} + \frac{\bar{\omega}^4x^2}{4} - \frac{i\bar{\omega}^5x^3}{12}\right) + \right.$$

$$\left. x^3\left(-\frac{i\bar{\omega}^3}{6} - \frac{\bar{\omega}^4x}{6} + \frac{i\bar{\omega}^5x^2}{12}\right) + x^4\left(\frac{\bar{\omega}^4}{24} - \frac{i\bar{\omega}^5x}{24}\right) + x^5\left(\frac{i\bar{\omega}^5}{120}\right)\right] = V\alpha \tag{14}$$

Next consider the normal velocity at $z=0$ associated with ϕ_1, namely

$$w_1=\left(\frac{\partial\phi_1}{\partial z}\right)_{z=0}=\frac{V\alpha}{\pi}\lim_{z\to0}\frac{\partial^2}{\partial z^2}\sum_{n=0}^{\infty}\left[a_{n1}\int\!\!\int_r D_n(\xi,\eta)\frac{\xi^{n+1}}{R}d\xi\,d\eta+\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M}\right)^{2m-2}a_{n1}\int\!\!\int_r D_n(\xi,\eta)\xi^{n+1}R^{2m-3}d\xi\,d\eta\right]\quad(15)$$

Examination of this equation and equation (13) reveals that the first integral expression on the right side of equation (15) yields $V\alpha$ exactly, provided that the distribution functions are determined so that the following integral equation is satisfied:

$$\frac{V\alpha}{\pi}\lim_{z\to0}\frac{\partial^2}{\partial z^2}\int\!\!\int_r D_n(\xi,\eta)\frac{\xi^{n+1}}{R}d\xi\,d\eta=V\alpha x^n\qquad(16)$$

The kernel of this integral equation has the form of a steady-state doublet potential. The problem of determining the distribution functions for this case is therefore analogous to determining distribution functions for certain steady-state problems. The distribution functions for steady-state problems, at least for those involving conical flow, can be determined by a straight-forward process. The main details of this process are given in the appendix, where a method of solving equation (16) for a triangular wing is derived and the distribution functions required to derive the velocity potential for this wing to the fifth power of $\bar{\omega}$ are given. It is to be noted that the method derived for solving equation (16) for a triangular wing may be generalized to apply to various plan forms and to problems of satisfying the boundary conditions for various velocity distributions.

From this point on, the analysis is restricted to the derivation of terms of the expanded potential involving $\bar{\omega}$ to the fifth power. The method of deriving these first few terms is quite general and can be used to obtain as many additional terms of the expanded potential as may be desired. As previously pointed out, however, terms of the potential after the first few become very unwieldy.

If the appropriate distribution functions are known for terms involving $\bar{\omega}$ to the fifth power, equation (15) may be written as follows:

$$w_1=V\alpha+w_2\qquad(17)$$

where

$$w_2=-\frac{V\alpha}{\pi}\lim_{z\to0}\frac{\partial^2}{\partial z^2}\frac{\bar{\omega}^2}{2M^2}\int\!\!\int_r\left[\left(1-i\bar{\omega}x-\frac{\bar{\omega}^2x^2}{2}+\frac{i\bar{\omega}^3x^3}{6}\right)\xi D_0R+\left(i\bar{\omega}+\bar{\omega}^2x-\frac{i\bar{\omega}^3x^2}{2}\right)\xi^2D_1R-\left(\frac{\bar{\omega}^2}{2}-\frac{i\bar{\omega}^3x}{2}\right)\xi^3D_2R+\frac{i\bar{\omega}^3}{6}\xi^4D_3R-\right.$$
$$\left.\left(\frac{\bar{\omega}^2}{12M^2}-\frac{i\bar{\omega}^3x}{12M^2}\right)\xi D_0R^3-\frac{i\bar{\omega}^3}{12M^2}\xi^2D_1R^3\right]d\xi\,d\eta\qquad(18)$$

is the additional vertical velocity arising from the second integral expression in equation (12) involving $\bar{\omega}$ to the fifth power. In order to maintain the boundary condition for ϕ_α, this additional velocity w_2 must be canceled. As previously pointed out this canceling, to a required order, can be achieved by considering other doublet solutions to equation (1). For this particular case consider relations $\bar{\phi}_1$ and $\bar{\bar{\phi}}_1$ similar to ϕ_1 (equation (12)) having the following forms:

$$\bar{\phi}_1=\frac{V\alpha}{\pi}\frac{\partial}{\partial z}\frac{\bar{\omega}^2}{2M^2}\sum_{n=0}^{\infty}\left[a_{n1}\int\!\!\int_r\overline{D}_n(\xi,\eta)\frac{\xi^{n+3}}{R}d\xi\,d\eta+\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M}\right)^{2m-2}a_{n1}\int\!\!\int_r\overline{D}_n(\xi,\eta)\xi^{n+3}R^{2m-3}d\xi\,d\eta\right]\qquad(19a)$$

$$\bar{\bar{\phi}}_1=\frac{V\alpha}{\pi}\frac{\partial}{\partial z}\frac{\bar{\omega}^4}{4M^4}\sum_{n=0}^{\infty}\left[a_{n1}\int\!\!\int\overline{\overline{D}}_n(\xi,\eta)\frac{\xi^{n+5}}{R}d\xi\,d\eta+\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-2)!}\left(\frac{\bar{\omega}}{M}\right)^{2m-2}a_{n1}\int\!\!\int_r\overline{\overline{D}}_n(\xi,\eta)\xi^{n+5}R^{2m-3}d\xi\,d\eta\right]\qquad(19b)$$

The vertical velocity distributions $\overline{w}_1$ and $\overline{w}_2$, to the fifth power of $\bar{\omega}$, arising from these expressions are

$$\overline{w}_1=\frac{V\alpha}{2\pi}\lim_{z\to0}\frac{\partial^2}{\partial z^2}\left(\frac{\bar{\omega}}{M}\right)^2\int\!\!\int_r\left[\left(1-i\bar{\omega}x-\frac{\bar{\omega}^2x^2}{2}+\frac{i\bar{\omega}^3x^3}{6}\right)\xi^3\overline{D}_0\frac{1}{R}+\left(i\bar{\omega}+\bar{\omega}^2x-\frac{i\bar{\omega}^3x^2}{2}\right)\xi^4\overline{D}_1\frac{1}{R}-\left(\frac{\bar{\omega}^2}{2}-\frac{i\bar{\omega}^3x}{2}\right)\xi^5\overline{D}_2\frac{1}{R}-\right.$$
$$\left.\frac{i\bar{\omega}^3}{6}\xi^6\overline{D}_3\frac{1}{R}-\frac{\bar{\omega}^2}{2M^2}(1-i\bar{\omega}x)\xi^3\overline{D}_0R-\frac{i\bar{\omega}^3}{2M^2}\xi^4\overline{D}_1R\right]d\xi\,d\eta\qquad(20a)$$

$$\overline{w}_2=\frac{V\alpha}{4\pi}\lim_{z\to0}\frac{\partial^2}{\partial z^2}\left(\frac{\bar{\omega}}{M}\right)^4\int\!\!\int_r\left[(1-i\bar{\omega}x)\xi^5\overline{\overline{D}}_0\frac{1}{R}+i\bar{\omega}\xi^6\overline{\overline{D}}_1\frac{1}{R}\right]d\xi\,d\eta\qquad(20b)$$

In these equations the distribution functions $\overline{D}_0$, $\overline{D}_1$, and so forth can be determined, as discussed subsequently, by the method given in the appendix so that $\overline{w}_1+\overline{w}_2$ is identically equal in value but opposite in sign to w_2. When these functions are determined, the boundary condition for ϕ_α is satisfied by $w_1+\overline{w}_1+\overline{w}_2=V\alpha$, which implies that the potential ϕ_α to the

fifth power of $\bar{\omega}$ is given by the sum

$$\phi_a=\phi_1+\bar{\phi}_1+\bar{\bar{\phi}}_1 \tag{21}$$

to this power of $\bar{\omega}$.

Expressions for $\bar{D}_n$ ($n=0, 1, 2, 3$) and $\bar{\bar{D}}_n$ ($n=0, 1$) are given together with expressions for other distribution functions D_n ($n=0, 1, 2, 3, 4, 5$) in the appendix. In regard to the determination of $\bar{D}_n$ and $\bar{\bar{D}}_n$, it may appear necessary, in order to formulate integral equations for these functions, to perform the generally unwieldy integrations of the type

$$\lim_{z\to 0}\frac{\partial^2}{\partial z^2}\int_r\!\!\int D_n(\xi,\eta)\xi^{n+1}R^{2m-3}\,d\xi\,d\eta \tag{22}$$

appearing in equation (15). In general, however, the information necessary for the determination of the functions $\bar{D}_n$ and $\bar{\bar{D}}_n$ can be obtained, as is done in the derivation of the functions D_n in the appendix, by examining the values of these integrals and their derivatives with regard to the parameter $\theta=\frac{y}{x}$ at some particular value of θ.

Returning to equation (21) and introducing into this equation the expression for ϕ_1 (equation (12)) and the expressions for $\bar{\phi}_1$ and $\bar{\bar{\phi}}_1$ (equations (19)), each to the fifth power of $\bar{\omega}$, gives for the potential ϕ_a to this power of $\bar{\omega}$ the following result:

$$\phi_a=\frac{V\alpha}{\pi}\lim_{z\to 0}\frac{\partial}{\partial z}\int_r\!\!\int\left\{\xi D_0\left[\left(1-i\bar{\omega}x-\frac{\bar{\omega}^2x^2}{2}+\frac{i\bar{\omega}^3x^3}{6}+\frac{\bar{\omega}^4x^4}{24}-\frac{i\bar{\omega}^5x^5}{120}\right)\frac{1}{R}-\left(\frac{\bar{\omega}^2}{2M^2}-\frac{i\bar{\omega}^3x}{2M^2}-\frac{\bar{\omega}^4x^2}{4M^2}+\frac{i\bar{\omega}^5x^3}{12M^2}\right)R+\left(\frac{\bar{\omega}^4}{24M^4}-\frac{i\bar{\omega}^5x}{24M^4}\right)R^3\right]+\right.$$
$$\xi^2D_1\left[\left(i\bar{\omega}+\bar{\omega}^2x-\frac{i\bar{\omega}^3x^2}{2}-\frac{\bar{\omega}^4x^3}{6}+\frac{i\bar{\omega}^5x^4}{24}\right)\frac{1}{R}-\left(\frac{i\bar{\omega}^3}{2M^2}+\frac{\bar{\omega}^4x}{2M^2}-\frac{i\bar{\omega}^5}{4M^2}\right)R+\frac{i\bar{\omega}^5}{24M^4}R^3\right]-\xi^3D_2\left[\left(\frac{\bar{\omega}^2}{2}-\frac{i\bar{\omega}^3x}{2}-\frac{\bar{\omega}^4x^2}{4}+\frac{i\bar{\omega}^5x^3}{12}\right)\frac{1}{R}-\right.$$
$$\left.\left(\frac{\bar{\omega}^4}{4M^2}-\frac{i\bar{\omega}^5x}{4M^2}\right)R\right]-\xi^4D_3\left[\left(\frac{i\bar{\omega}^3}{6}+\frac{\bar{\omega}^4x}{6}-\frac{i\bar{\omega}^5x^2}{12}\right)\frac{1}{R}+\frac{i\bar{\omega}^5}{12M^2}R\right]+\xi^5D_4\left[\left(\frac{\bar{\omega}^4}{24}-\frac{i\bar{\omega}^5x}{24}\right)\frac{1}{R}\right]+\xi^6D_5\left[\left(\frac{i\bar{\omega}^5}{120}\right)\frac{1}{R}\right]+$$
$$\frac{\bar{\omega}^2}{2M^2}\xi^3\bar{D}_0\left[\left(1-i\bar{\omega}x-\frac{\bar{\omega}^2x^2}{2}+\frac{i\bar{\omega}^3x^3}{6}\right)\frac{1}{R}-\left(\frac{\bar{\omega}^2}{2M^2}-\frac{i\bar{\omega}^3x}{2M^2}\right)R\right]+\frac{\bar{\omega}^2}{2M^2}\xi^4\bar{D}_1\left[\left(i\bar{\omega}+\bar{\omega}^2x-\frac{i\bar{\omega}^3x^2}{2}\right)\frac{1}{R}-\frac{i\bar{\omega}^3}{2M^2}R\right]-$$
$$\frac{\bar{\omega}^2}{2M^2}\xi^5\bar{D}_2\left[\left(\frac{\bar{\omega}^2}{2}-\frac{i\bar{\omega}^3x}{2}\right)\frac{1}{R}\right]-\frac{\bar{\omega}^2}{2M^2}\xi^6\bar{D}_3\left[\left(\frac{i\bar{\omega}^3}{6}\right)\frac{1}{R}\right]+\frac{\bar{\omega}^4}{4M^4}\xi^5\bar{\bar{D}}_0\left[(1-i\bar{\omega}x)\frac{1}{R}\right]+\frac{\bar{\omega}^4}{4M^4}\xi^6\bar{\bar{D}}_1\left[(i\bar{\omega})\frac{1}{R}\right]\right\}\,d\xi\,d\eta \tag{23}$$

Since it may be shown, as in reference 8, for example, that

$$\frac{V\alpha}{\pi}\lim_{z\to 0}\frac{\partial}{\partial z}\int_r\!\!\int\frac{D_n\xi^{n+1}}{R}\,d\xi\,d\eta=V\alpha x^n D_n(x,y)$$

equation (23) reduces, at $z=0$, to

$$\phi_a=V\alpha\left[\left(1-i\bar{\omega}x-\frac{\bar{\omega}^2x^2}{2}+\frac{i\bar{\omega}^3x^3}{6}+\frac{\bar{\omega}^4x^4}{24}-\frac{i\bar{\omega}^5x^5}{120}\right)D_0(x,y)+x\left(i\bar{\omega}+\bar{\omega}^2x-\frac{i\bar{\omega}^3x^2}{2}-\frac{\bar{\omega}^4x^3}{6}+\frac{i\bar{\omega}^5x^4}{24}\right)D_1(x,y)-x^2\left(\frac{\bar{\omega}^2}{2}-\frac{i\bar{\omega}^3x}{2}-\right.\right.$$
$$\left.\frac{\bar{\omega}^4x^2}{4}+\frac{i\bar{\omega}^5x^3}{12}\right)D_2(x,y)-x^3\left(\frac{i\bar{\omega}^3}{6}+\frac{\bar{\omega}^4x}{6}-\frac{i\bar{\omega}^5x^2}{12}\right)D_3(x,y)+x^4\left(\frac{\bar{\omega}^4}{24}-\frac{i\bar{\omega}^5x}{24}\right)D_4(x,y)+\frac{i\bar{\omega}^5x^5}{120}D_5(x,y)+x^3\left(\frac{\bar{\omega}^2}{2M^2}-\frac{i\bar{\omega}^3x}{2M^2}-\right.$$
$$\left.\frac{\bar{\omega}^4x^2}{4M^2}+\frac{i\bar{\omega}^5x^3}{12M^2}\right)\bar{D}_0(x,y)+x^3\left(\frac{i\bar{\omega}^3}{2M^2}+\frac{\bar{\omega}^4x}{2M^2}-\frac{i\bar{\omega}^5x^2}{4M^2}\right)\bar{D}_1(x,y)-x^4\left(\frac{\bar{\omega}^4}{4M^2}-\frac{i\bar{\omega}^5x}{4M^2}\right)\bar{D}_2-\frac{i\bar{\omega}^5x^5}{12M^2}\bar{D}_3(x,y)+$$
$$x^4\left(\frac{\bar{\omega}^4}{4M^4}-\frac{i\bar{\omega}^5x}{4M^4}\right)\bar{\bar{D}}_0(x,y)+\frac{i\bar{\omega}^5x^5}{4M^4}\bar{\bar{D}}_1(x,y)\right] \tag{24}$$

which, after the expressions for the distribution functions given in the appendix are substituted and the terms are regrouped, may be written in the following simple form:

$$\phi_a=V\alpha\sqrt{C^2x^2-y^2}\,[A_0-i\bar{\omega}x\sigma_1-\bar{\omega}^2(\sigma_3x^2+\sigma_4\beta^2y^2)+i\bar{\omega}^3(\sigma_5x^3+\sigma_6\beta^2y^2x)+\bar{\omega}^4(\sigma_9x^4+\sigma_{10}\beta^2y^2x^2+\sigma_{11}\beta^4y^4)-i\bar{\omega}^5(\sigma_{12}x^5+$$
$$\sigma_{13}\beta^2y^2x^3+\sigma_{14}\beta^4y^4x)] \tag{25}$$

The quantities A_0, σ_1, and so forth in this equation are part of a group of quantities A_i and $\sigma_j = P_j + \dfrac{Q_j}{M^2} + \dfrac{R_j}{M^4}$ which are defined in the appendix; these quantities are suitable for writing the potential and expressions subsequently derived for forces and moments in simple form. The quantities A_i, P_j, Q_j, and R_j are functions only of the product βC (ratio of tangent of the half apex angle of the triangle to the Mach angle) and are tabulated for some values of βC in table I and are shown plotted in figures 3 and 4, respectively. The quantities σ_j, which are functions of Mach number M and the product βC, may be evaluated for particular values of M from the values of P_j, Q_j, and R_j in table I or from the plots in figure 4.

The quantities A_0 and A_1 are the same, as should be expected, as the parameters associated, respectively, with constant angle of attack and constant pitching of triangular wings of references 9 and 10.

Expressions for the potentials $\phi_{\dot{\alpha}}$ and $\phi_{\dot{h}}$ can be obtained by the method discussed for obtaining ϕ_α, or they can be obtained to the fifth order of $\bar{\omega}$ by comparison and synthesis from equation (23). After simplification these expressions are

$$\phi_{\dot{\alpha}} = \dot{\alpha}\left\{ \sqrt{C^2 x^2 - y^2}\, [A_1 x - i\bar{\omega}(\sigma_2 x^2 - A_3 \beta^2 y^2) - \bar{\omega}^2(\sigma_7 x^3 - \sigma_8 \beta^2 y^2 x) + \right.$$
$$i\bar{\omega}^3(\sigma_{15} x^4 + \sigma_{16}\beta^2 y^2 x^2 + \sigma_{17}\beta^4 y^4) + \bar{\omega}^4(\sigma_{18} x^5 + \sigma_{19}\beta^2 y^2 x^3 +$$
$$\left. \sigma_{20}\beta^4 y^4 x)] - \frac{x_0}{h}\,\phi_{\dot{h}} \right\} \tag{26}$$

$$\phi_{\dot{h}} = \dot{h}\sqrt{C^2 x^2 - y^2}\,[A_0 - i\bar{\omega} x \sigma_1 - \bar{\omega}^2(\sigma_3 x^2 + \sigma_4 \beta^2 y^2) +$$
$$i\bar{\omega}^3(\sigma_5 x^3 + \sigma_6 \beta^2 y^2 x) + \bar{\omega}^4(\sigma_9 x^4 + \sigma_{10}\beta^2 y^2 x^2 + \sigma_{11}\beta^4 y^4)] \tag{27}$$
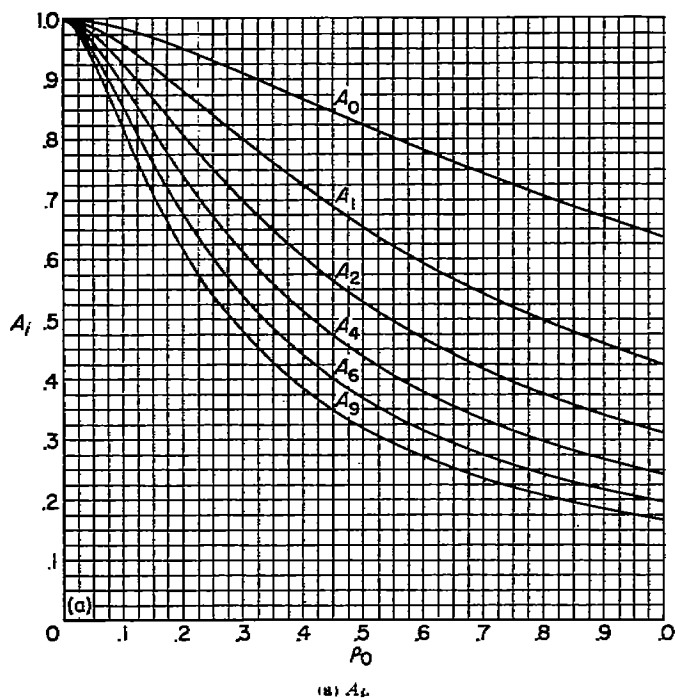
At $\beta C = 1$ or $C = \dfrac{1}{\beta}$, which is the condition at which the Mach lines from the apex of the triangle coincide with the leading edges of the triangle, equations (23), (26), and (27) reduce, respectively, to

$$(\phi_\alpha)_{\beta C = 1} = \frac{2 V\alpha}{\beta\pi}\sqrt{x^2 - \beta^2 y^2}\left\{ 1 - \frac{i\bar{\omega}x}{3} - \frac{\bar{\omega}^2}{90 M^2}\,[(5 + 7 M^2)x^2 - \right.$$
$$(5 - 2M^2)\beta^2 y^2] + \frac{i\bar{\omega}^3}{210 M^2}\,[(7 + 3M^2)x^3 -$$
$$(7 - 2M^2)\beta^2 y^2 x] + \frac{\bar{\omega}^4}{37800 M^4}\,[(83 M^4 + 414 M^2 + 63)x^4 +$$
$$42(2M^4 - 9M^2 - 3)\beta^2 y^2 x^2 + (8M^4 - 36M^2 + 63)\beta^4 y^4] -$$
$$\frac{i\bar{\omega}^5}{249480 M^4}\,[(73 M^4 + 638 M^2 + 297)x^5 + 2(46 M^4 -$$
$$\left. 253 M^2 - 297)\beta^2 y^2 x^3 + 3(8M^4 - 44 M^2 + 99)\beta^4 y^4 x] \right\}$$
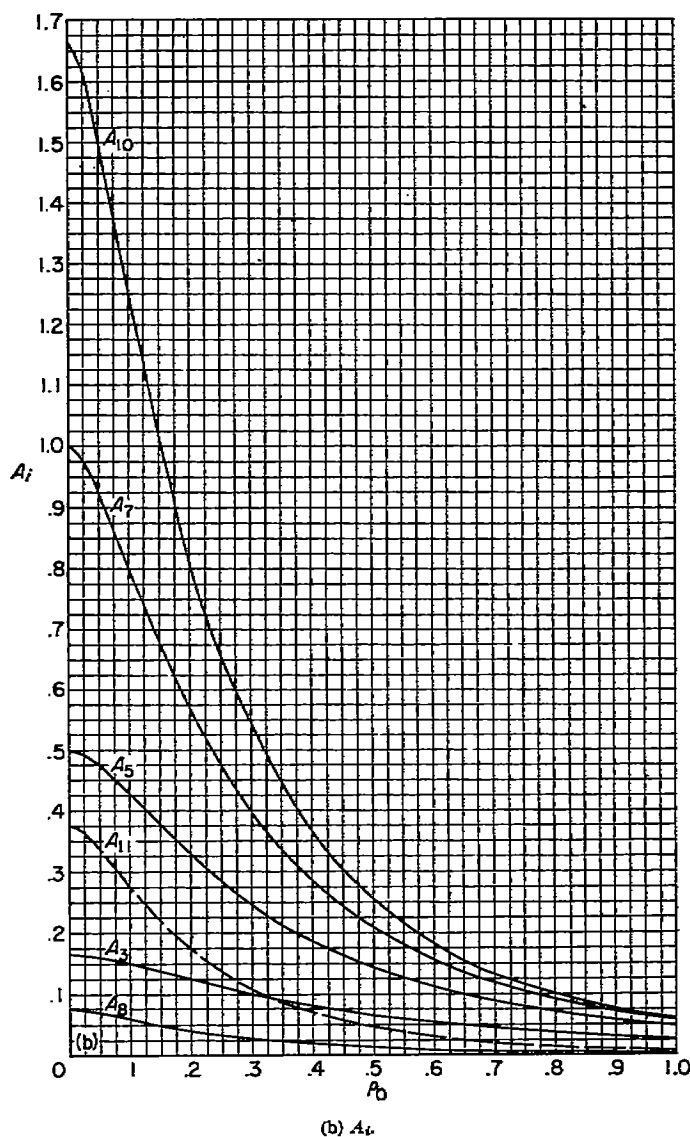$$\tag{28}$$

(a) A_i.

(b) $\bar{A}_i$.

FIGURE 3.—Continued.

$$(\phi_{\Delta})_{\beta C=1}=\frac{4\dot{\alpha}\sqrt{x^2-\beta^2 y^2}}{3\beta\pi}\left\{x-\frac{i\bar{\omega}}{15}(4x^2-\beta^2 y^2)-\right.$$

$$\frac{\bar{\omega}^2}{210M^2}[(11M^2+7)x^3-(2M^2+7)\beta^2 y^2 x]+$$

$$\frac{i\bar{\omega}^3}{3150M^2}[2(13M^2+27)x^4+3(M^2-21)\beta^2 y^2 x^2-$$

$$(4M^2-9)\beta^4 y^4]+\frac{\bar{\omega}^4}{415800M^4}[(457M^4+2046M^2+$$

$$297)x^5+2(118M^4-1221M^2-297)\beta^2 y^2 x^3-$$

$$\left.(168M^4-396M^2-297)\beta^4 y^4 x]\right\}-\frac{x_0\dot{\alpha}}{h}(\phi_h)_{\beta C=1} \quad (29)$$
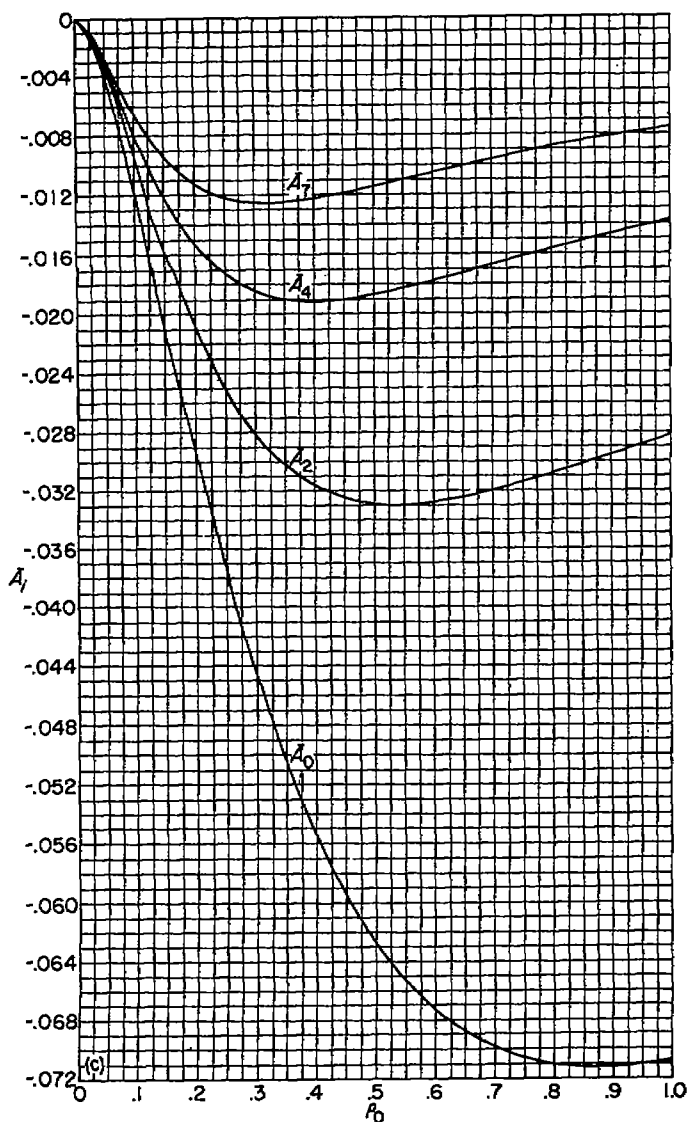
$$(\phi_h)_{\beta C=1}=\frac{2\dot{h}}{\beta\pi}\sqrt{x^2-\beta^2 y^2}\left\{1-\frac{i\bar{\omega}x}{3}-\frac{\bar{\omega}^2}{90M^2}[(5+7M^2)x^2-\right.$$

$$(5-2M^2)\beta^2 y^2]+\frac{i\bar{\omega}^3}{210M^2}[(7+3M^2)x^3-$$

$$(7-2M^2)\beta^2 y^2 x]+\frac{\bar{\omega}^4}{37800M^4}[(83M^4+414M^2+63)x^4+$$

$$\left.42(2M^4-9M^2-3)\beta^2 y^2 x^2+(8M^4-36M^2+63)\beta^4 y^4]\right\}$$

$$(30)$$

Equations (28), (29), and (30) can also be obtained by going to the limit $\beta C=1$ in the expanded potential for triangular wings with supersonic leading edges (see, for example, reference 2 for results to the third power of $\bar{\omega}$) or by integrating

(c) $\bar{A}_i$.
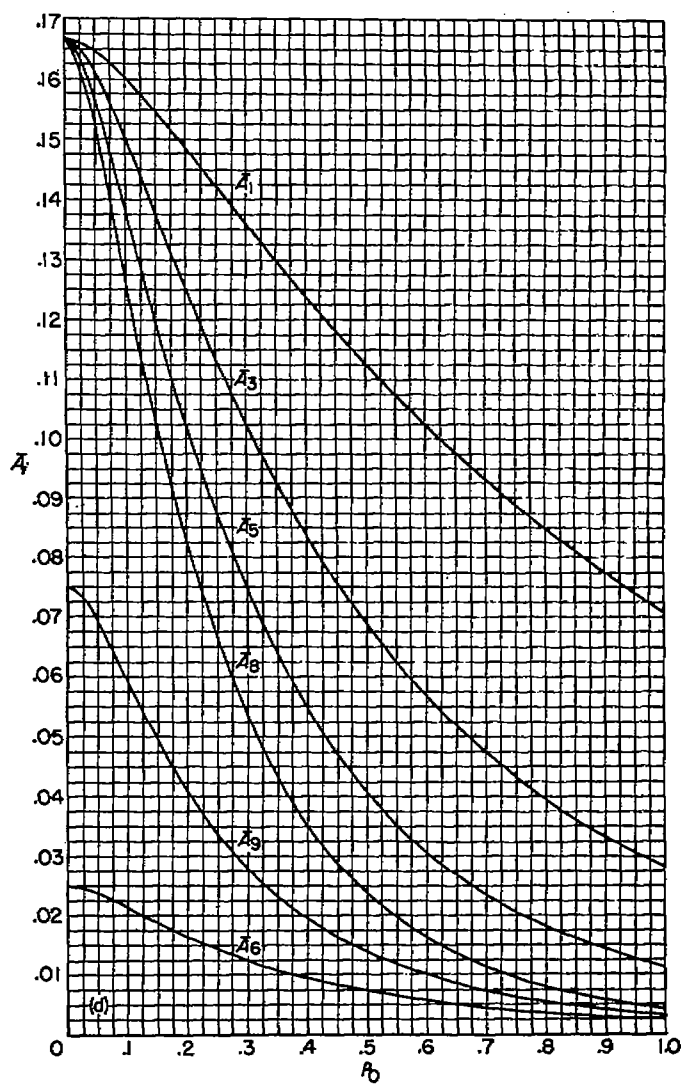
Figure 3.—Continued.

(d) $\bar{A}_i$.

Figure 3.—Continued.

the expanded potential of a unit source over the region occupied by the wing with $\beta C=1$. Equations (28), (29), and (30) therefore serve as a check on the results given in equations (25), (26), and (27), respectively.

For values of the product βC such that $\beta^2 C^2 \ll 1$, equations (25), (26), and (27) reduce, respectively, to

$$\phi_\alpha = V\alpha\sqrt{C^2 x^2 - y^2} \tag{31}$$

$$\phi_{\dot\alpha} = \dot\alpha\sqrt{C^2 x^2 - y^2}\,(x - x_0) \tag{32}$$

$$\phi_{\dot h} = \dot h\sqrt{C^2 x^2 - y^2} \tag{33}$$

These expressions are the counterpart of the potential associated with triangular wings of vanishingly small aspect ratio in steady flow (reference 11) and may thus be regarded as the potential associated with a small-aspect-ratio triangle oscillating in pitch and vertical translation in either subsonic or supersonic flow, provided that $\bar\omega$ remains finite. The condition $\beta C \ll 1$ is obviously satisfied for all values of C when $M=1$. However, as $M \to 1$, the value of $\bar\omega$ becomes infinite and the expanded potential, as treated herein, becomes meaningless.

FORCES AND MOMENTS

As pointed out in the introduction, the velocity potential for the triangular wing can be used to calculate the aerodynamic forces and moments for other plan forms that can be formed from the triangular wing by cutting the trailing edges so that they lie ahead of the Mach cones emanating from their foremost points. Sketches of different plan forms thus obtained are shown in figure 5.

The force and moment coefficients desirable for most flutter calculations are those that yield the spanwise variation in these quantities or chordwise force and moment coefficients. These coefficients are obtained by integrating the pressure difference along any chord for the forces and the pressure difference multiplied by a moment arm for the moments. A convenient procedure in deriving these quantities is to introduce the reduced frequency parameter $\frac{b\omega}{V}=k$ and to employ the variables x, y, and x_0 in a new sense as nondimensional quantities obtained by dividing the old variables by the maximum chord $2b$ of the wing.

The pressure difference between the upper and lower surfaces of the wing is

$$\Delta p = -2\rho\left(\frac{V}{2b}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t}\right) \tag{34}$$

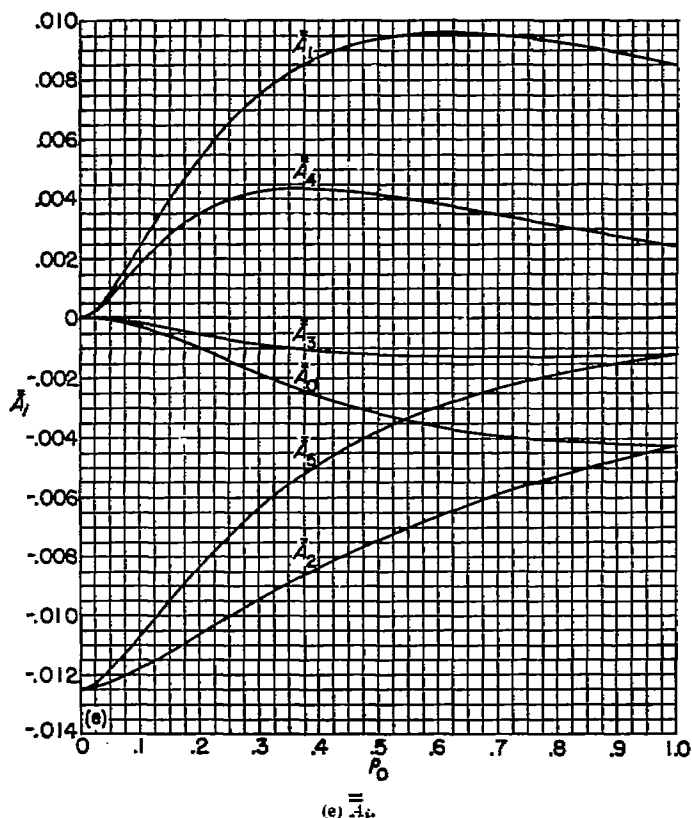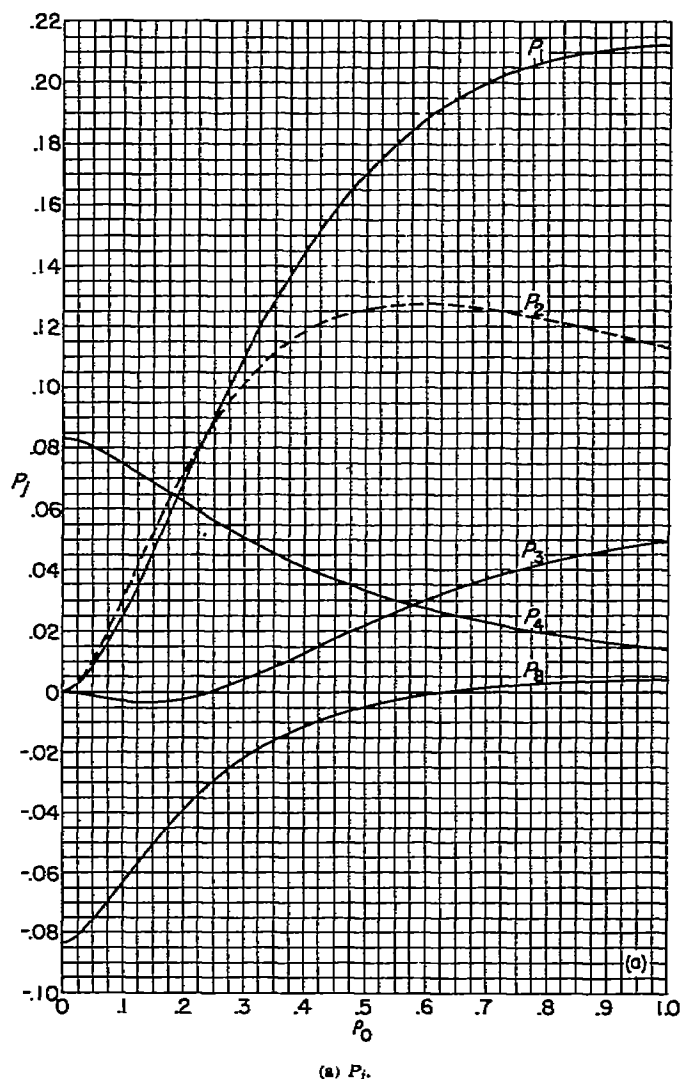(e) $\frac{\bar\equiv}{A_i}$

FIGURE 3.—Concluded.

(a) P_i.

FIGURE 4.—Variation of the quantities P_i, Q_i, and R_i as functions of ρ_0

(b) P_j.

Figure 4.—Continued.

(c) P_j.

Figure 4.—Continued.

(d) P_j.
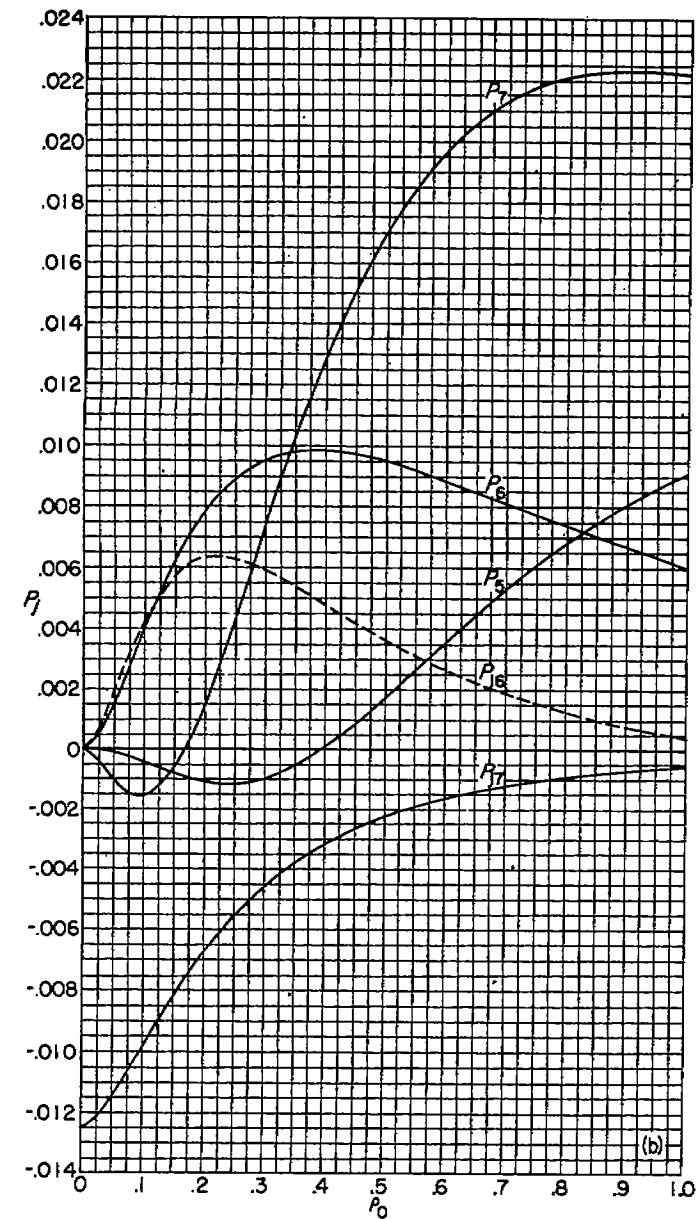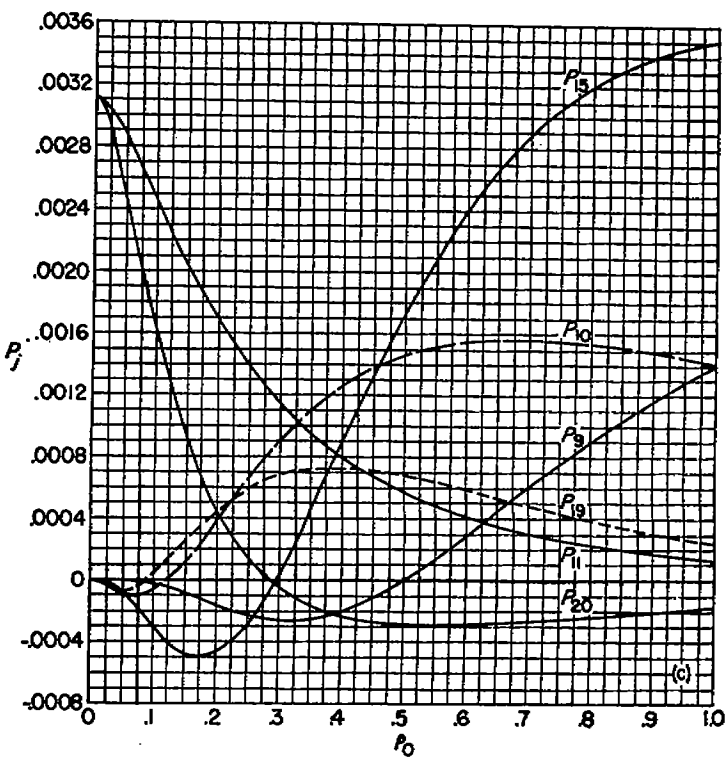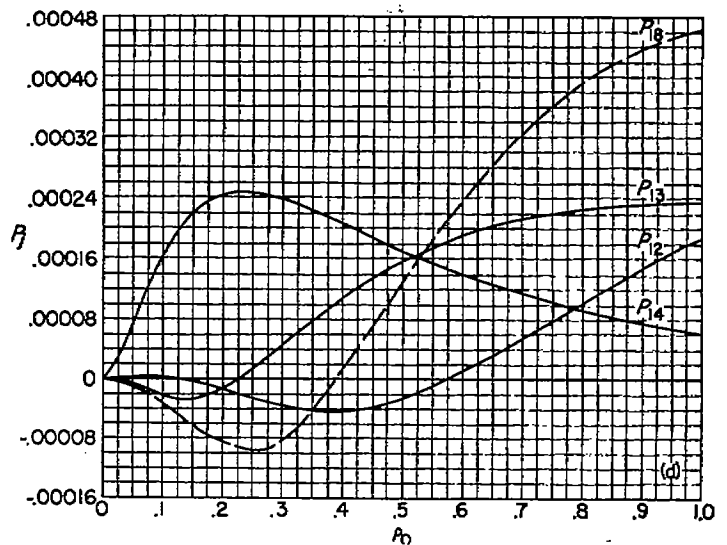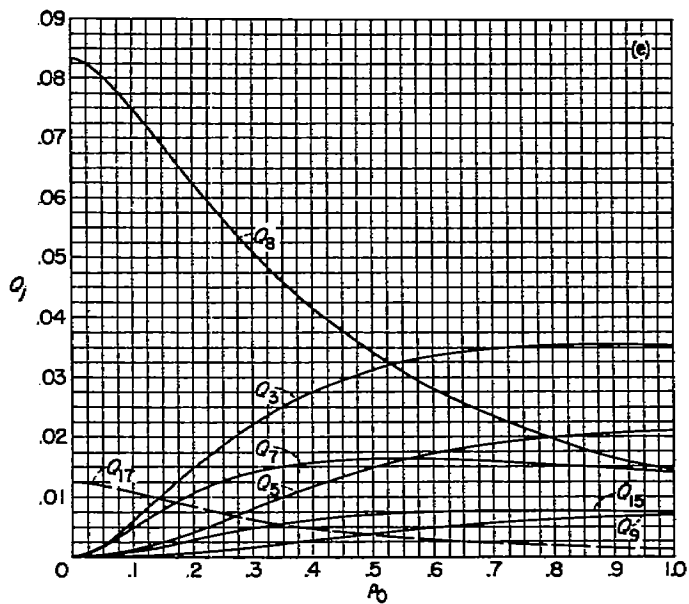
Figure 4.—Continued.

(e) Q_j.

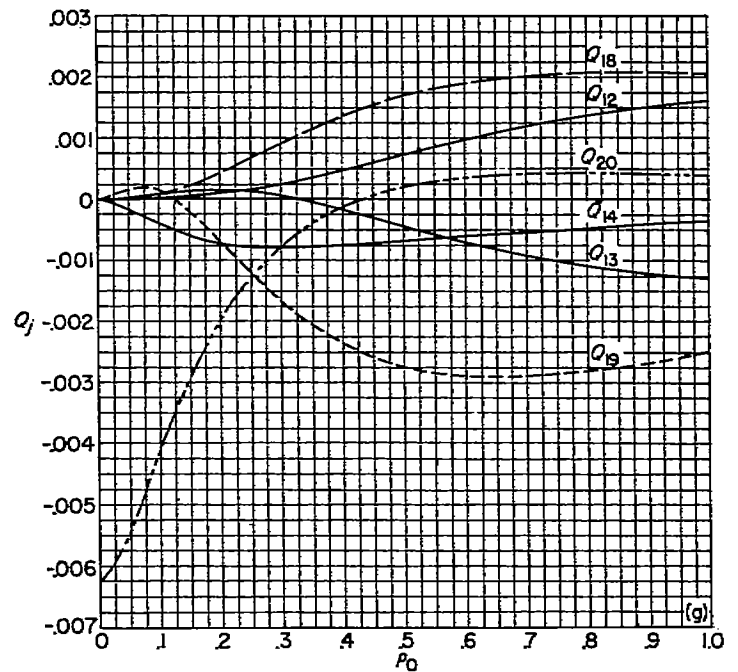FIGURE 4.—Continued.

(g) Q_j.
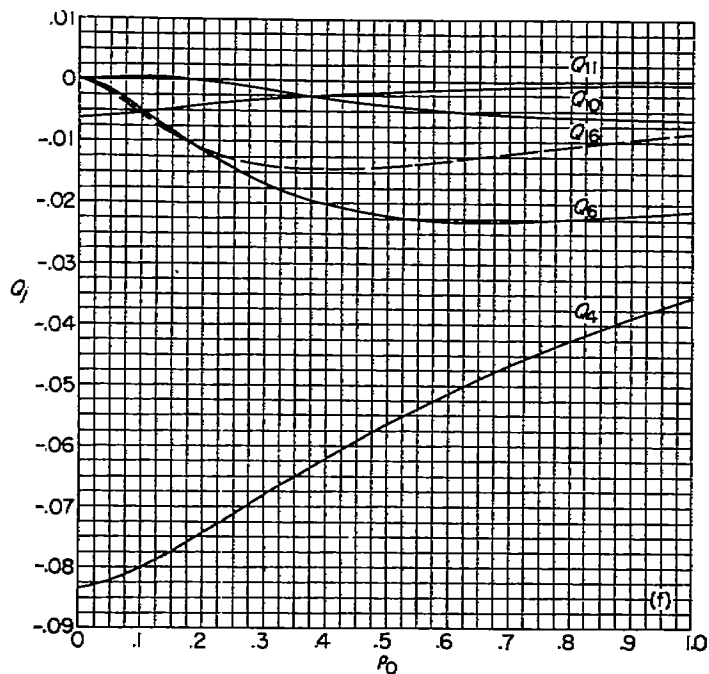
FIGURE 4.—Continued.

(f) Q_j.

FIGURE 4.—Continued.

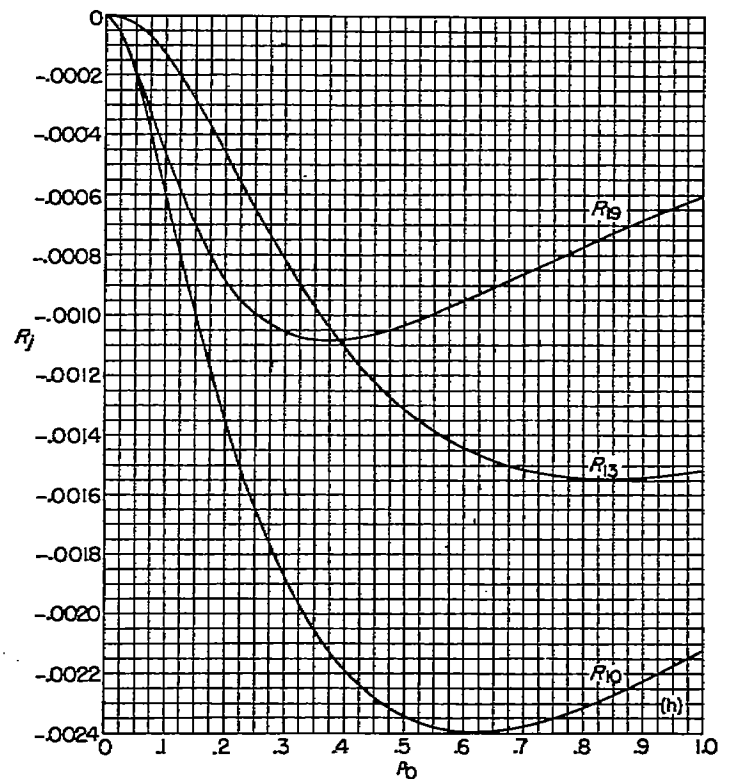

(h) R_j.

FIGURE 4.—Continued

The force, positive downward, at any section of any of the plan forms shown in figure 5 may be expressed as

$$P=2b\int_{y/C}^{x_1}\Delta p\ dx=-4\rho b\int_{y/C}^{x_1}\left(\frac{V}{2b}\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial t}\right)dx$$

$$=-2\rho V\left\{\phi\Big]_{y/C}^{x_1}+2ik\int_{y/C}^{x_1}\phi\ dx\right\}\qquad(35)$$

where x_1 has the following values with respect to the different configurations shown in figure 5:

In plan form A

$$x_1=1\qquad\qquad(36a)$$

In plan form B

$$x_1=1-\frac{y}{m}\qquad\qquad(36b)$$

In plan form C

$$x_1=1+\frac{y}{m}\qquad\qquad(36c)$$

In plan form D

$$x_1=1+\frac{y}{m}\quad\text{for }0\leqq y\leqq m\left(\frac{s}{2bC}-1\right)\ \Bigg\}$$
$$x_1=\frac{s}{2bC}\quad\text{for }m\left(\frac{s}{2bC}-1\right)<y\leqq\frac{s}{2b}\ \Bigg\}\qquad(36d)$$

After the expression for ϕ, given by the sum of equations (25), (26), and (27), is substituted into equation (35) and the integration is performed, the results may be reduced to the form

$$P=-4\rho b V^2k^2e^{i\omega t}\left[\frac{h_0}{b}(L_1+iL_2)+\alpha_0(L_3+iL_4)\right]\qquad(37)$$

where

(i) R_j.

FIGURE 4.—Concluded.

$$L_1=\sqrt{C^2x_1{}^2-y^2}\left[\frac{x_1}{\beta^2}(2M^2\sigma_1-\beta^2A_0)-\frac{2M^4k^2x_1{}^3}{\beta^6}(4M^2\sigma_5-\beta^2\sigma_3)-\frac{M^4k^2y^2x_1}{\beta^4C^2}(8M^2C^2\sigma_6+\sigma_3-4\beta^2C^2\sigma_4)\right]+$$

$$y^2\cosh^{-1}\frac{Cx_1}{y}\left[\frac{A_0}{C}-\frac{M^4k^2y^2}{\beta^4C^3}(\sigma_3+4\beta^2C^2\sigma_4)\right]\qquad(38a)$$

$$L_2=\sqrt{C^2x_1{}^2-y^2}\left[\frac{A_0}{k}-\frac{4M^2kx_1{}^2}{3\beta^4}(3M^2\sigma_3-\beta^2\sigma_1)-\frac{4M^2ky^2}{3\beta^2C^2}(3M^2C^2\sigma_4+\sigma_1)+\frac{16M^6k^3x_1{}^4}{5\beta^8}(5M^2\sigma_9-\beta^2\sigma_5)+\right.$$

$$\frac{16M^6k^3x_1{}^2y^2}{15\beta^6C^2}(15M^2C^2\sigma_{10}+\sigma_5-5\beta^2C^2\sigma_6)+\frac{16M^6k^3y^4}{15\beta^6C^4}(15M^2\beta^2C^4\sigma_{11}+2\sigma_5+5\beta^2C^2\sigma_6)\Big]\qquad(38b)$$

$$L_3=\sqrt{C^2x_1{}^2-y^2}\left[\frac{A_0}{k^2}+\frac{4x_1{}^2}{3\beta^4}(M^2\beta^2\sigma_1-3M^4\sigma_3+3M^2\beta^2\sigma_2-\beta^4A_1)-\frac{4y^2}{3\beta^2C^2}(3M^4C^2\sigma_4+M^2\sigma_1+3M^2\beta^2C^2A_3-\beta^2A_1)+\right.$$

$$\frac{16M^4k^2x_1{}^4}{5\beta^8}(5M^4\sigma_9-M^2\beta^2\sigma_5-5M^2\beta^2\sigma_{15}+\beta^4\sigma_7)+\frac{16M^4k^2x_1{}^2y^2}{15\beta^6C^2}(15M^4C^2\sigma_{10}+M^2\sigma_5-5M^2\beta^2C^2\sigma_6-15M^2\beta^2C^2\sigma_{16}-$$

$$\beta^2\sigma_7-5\beta^4C^2\sigma_8)+\frac{16M^4k^2y^4}{15\beta^6C^4}(15M^4\beta^2C^4\sigma_{11}+2M^2\sigma_5+5M^2\beta^2C^2\sigma_6-15M^2\beta^4C^4\sigma_{17}-2\beta^2\sigma_7+5\beta^4C^2\sigma_8)\Big]-2x_0L_1\qquad(38c)$$

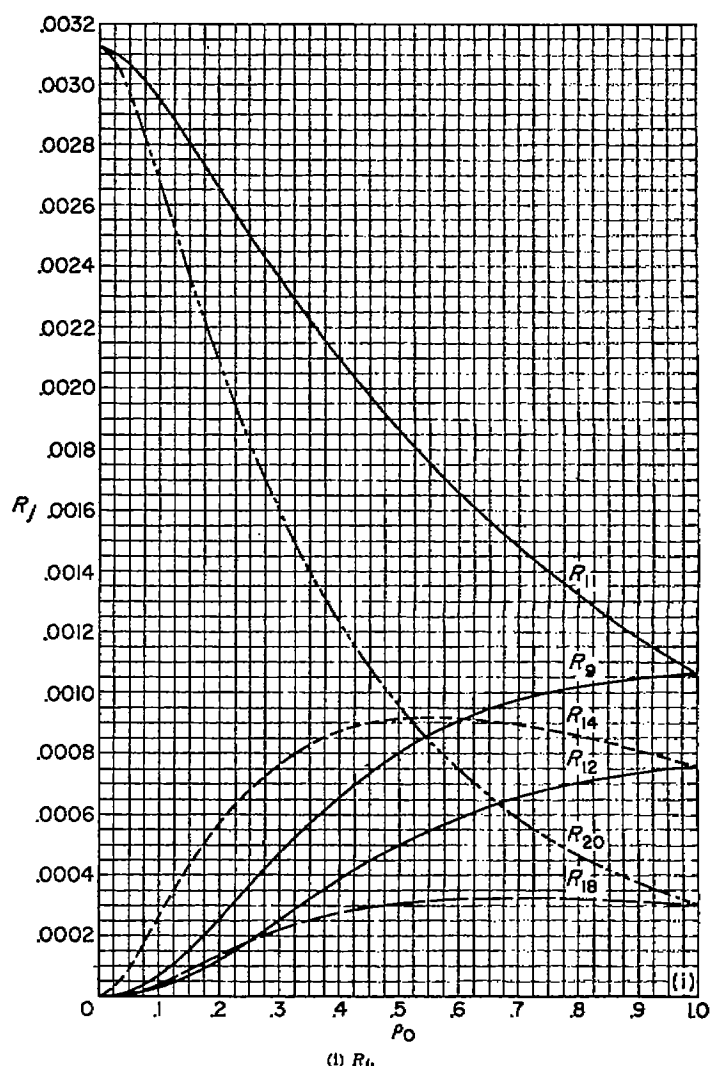$$L_4=\sqrt{C^2x_1^2-y^2}\left[\frac{x_1}{k\beta^2}(\beta^2A_0+2\beta^2A_1-2M^2\sigma_1)+\frac{2M^2kx_1^3}{\beta^6}(4M^4\sigma_5-M^2\beta^2\sigma_2-4M^2\beta^2\sigma_7+\beta^4\sigma_2)+\frac{M^2kx_1y^2}{\beta^4C^2}(8M^4C^2\sigma_6+M^2\sigma_2-\right.$$

$$4M^2\beta^2C^2\sigma_4+8M^2\beta^2C^2\sigma_8-\beta^2\sigma_2-4\beta^4C^2A_3)-\frac{16M^6k^3x_1^5}{3\beta^{10}}(6M^4\sigma_{12}-M^2\beta^2\sigma_9-6M^2\beta^2\sigma_{18}+\beta^4\sigma_{15})-\frac{M^6k^3x_1^3y^2}{3\beta^8C^2}(96M^4C^2\sigma_{13}+$$

$$4M^2\sigma_9-24M^2\beta^2C^2\sigma_{10}-96M^2\beta^2C^2\sigma_{19}-4\beta^2\sigma_{15}+24\beta^4C^2\sigma_{16})-\frac{M^6k^3x_1y^4}{\beta^8C^4}(32M^4\beta^2C^4\sigma_{14}+2M^2\sigma_9+4M^2\beta^2C^2\sigma_{10}-$$

$$16M^2\beta^4C^4\sigma_{11}-32M^2\beta^4C^4\sigma_{20}-2\beta^2\sigma_{15}-4\beta^4C^2\sigma_{16}+16\beta^6C^4\sigma_{17})\right]-y^2\cosh^{-1}\frac{Cx_1}{y}\left[\frac{A_0}{Ck}-\frac{M^2ky^2}{\beta^4C^3}(M^2\sigma_3+4M^2\beta^2C^2\sigma_4-\right.$$

$$\beta^2\sigma_2+4\beta^4C^2A_3)+\frac{M^6k^3y^4}{\beta^8C^5}(2M^2\sigma_9+4M^2\beta^2C^2\sigma_{10}+16M^2\beta^4C^4\sigma_{11}-2\beta^2\sigma_{15}-4\beta^4C^2\sigma_{16}-16\beta^6C^4\sigma_{17})\right]-2x_0L_2 \tag{38d}$$

In a similar manner, the moment (positive leading edge up) about the axis $x=x_0$ is

$$M_\alpha=-4b^2\int_{y/C}^{x_1}(x-x_0)\Delta p\,dx$$
$$=-4\rho V^2k^2b^2\left[\frac{h_0}{b}(M_1+iM_2)+\alpha_0(M_3+iM_4)\right] \tag{39}$$

where

$$M_1=\sqrt{C^2x_1^2-y^2}\left[\frac{4x_1^2}{3\beta^2}(2M^2\sigma_1-\beta^2A_0)+\frac{4y^2}{3\beta^2C^2}(M^2\sigma_1+\beta^2A_0)+\frac{16M^4k^2x_1^4}{5\beta^6}(\beta^2\sigma_2-4M^2\sigma_6)-\frac{16M^4k^2x_1^2y^2}{15\beta^6C^2}(\beta^2\sigma_3-\right.$$

$$5\beta^4C^2\sigma_4+M^2\sigma_5+10M^2\beta^2C^2\sigma_6)-\frac{16M^4k^2y^4}{15\beta^6C^4}(2\beta^2\sigma_3+5\beta^4C^2\sigma_4+2M^2\sigma_5+5M^2\beta^2C^2\sigma_6)\right]-2x_0L_1 \tag{40a}$$

$$M_2=\sqrt{C^2x_1^2-y^2}\left[\frac{A_0x_1}{k}+\frac{2M^2kx_1^3}{\beta^4}(\beta^2\sigma_1-3M^2\sigma_2)-\frac{M^2kx_1y^2}{\beta^4C^2}(\beta^2\sigma_1+M^2\sigma_3+4M^2\beta^2C^2\sigma_4)-\frac{16M^6k^3x_1^5}{3\beta^8}(\beta^2\sigma_5-5M^2\sigma_9)+\right.$$

$$\frac{4M^6k^3x_1^3y^2}{3\beta^8C^2}(\beta^2\sigma_5-6\beta^4C^2\sigma_6+M^2\sigma_9+18M^2\beta^2C^2\sigma_{10})+\frac{2M^6k^3x_1y^4}{\beta^8C^4}(\beta^2\sigma_5+2\beta^4C^2\sigma_6+M^2\sigma_9+2M^2\beta^2C^2\sigma_{10}+$$

$$8M^2\beta^4C^4\sigma_{11})\right]+y^2\cosh^{-1}\frac{Cx_1}{y}\left[\frac{A_0}{Ck}-\frac{M^2ky^2}{\beta^4C^3}(M^2\sigma_3+4M^2\beta^3C^2\sigma_4+\beta^2\sigma_1)+\right.$$

$$\frac{2M^6k^3y^4}{\beta^8C^5}(M^2\sigma_9+2M^2\beta^2C^2\sigma_{10}+8M^2\beta^4C^4\sigma_{11}+\beta^2\sigma_5+2\beta^4C^2\sigma_6)\right]-2x_0L_2 \tag{40b}$$

$$M_3=\sqrt{C^2x_1^2-y^2}\left[\frac{A_0x_1}{k^2}+\frac{2x_1^3}{\beta^4}(M^2\beta^2\sigma_1-3M^4\sigma_3+3M^2\beta^2\sigma_2-\beta^4A_1)-\frac{x_1y^2}{\beta^4C^2}(M^2\beta^2\sigma_1+M^4\sigma_3+4M^4\beta^2C^2\sigma_4-\beta^4A_1-\right.$$

$$M^2\beta^2\sigma_2+4M^2\beta^4C^2A_3)-\frac{16M^4k^2x_1^5}{3\beta^8}(M^2\beta^2\sigma_5-5M^4\sigma_9-\beta^4\sigma_7+5M^2\beta^2\sigma_{18})+\frac{4M^4k^2x_1^3y^2}{3\beta^8C^2}(M^2\beta^2\sigma_5-6M^2\beta^4C^2\sigma_6+M^4\sigma_9+$$

$$18M^4\beta^2C^2\sigma_{10}-\beta^4\sigma_7-6\beta^6C^2\sigma_8-M^2\beta^2\sigma_{15}-18M^2\beta^4C^2\sigma_{16})+\frac{2M^4k^2x_1y^4}{\beta^8C^4}(M^2\beta^2\sigma_5+2M^2\beta^4C^2\sigma_6+M^4\sigma_9+2M^4\beta^2C^2\sigma_{10}+$$

$$8M^4\beta^4C^4\sigma_{11}-\beta^4\sigma_7+2\beta^6C^2\sigma_8-M^2\beta^2\sigma_{15}-2M^2\beta^4C^2\sigma_{16}-8M^2\beta^6C^4\sigma_{17})\right]+y^2\cosh^{-1}\frac{Cx_1}{y}\left[\frac{A_0}{Ck^2}-\frac{y^2}{\beta^4C^3}(M^2\beta^2\sigma_1+M^4\sigma_3+\right.$$

$$4M^4\beta^2C^2\sigma_4-\beta^4A_1-M^2\beta^2\sigma_2+4M^2\beta^4C^2A_3)+\frac{2M^4k^2y^4}{\beta^8C^5}(M^2\beta^2\sigma_5+2M^2\beta^4C^2\sigma_6+M^4\sigma_9+2M^4\beta^2C^2\sigma_{10}+8M^4\beta^4C^4\sigma_{11}-$$

$$\beta^4\sigma_7+2\beta^6C^2\sigma_8-M^2\beta^2\sigma_{15}-2M^2\beta^4C^2\sigma_{16}-8M^2\beta^6C^4\sigma_{17})\right]-2x_0(M_1+L_3+2x_0L_1) \tag{40c}$$

$$M_4 = \sqrt{C^2 x_1^2 - y^2} \left[\frac{4x_1^2}{3\beta^2 k} (\beta^2 A_0 - 2M^2 \sigma_1 + 2\beta^2 A_1) - \right.$$

$$\frac{4y^2}{3\beta^2 C^2 k} (M^2 \sigma_1 + \beta^2 A_0 - \beta^2 A_1) - \frac{16 M^2 k x_1^4}{5\beta^6} (M^2 \beta^2 \sigma_2 -$$

$$4M^4 \sigma_6 - \beta^4 \sigma_2 + 4M^2 \beta^2 \sigma_7) + \frac{16 M^2 k x_1^2 y^2}{15 \beta^6 C^2} (M^2 \beta^2 \sigma_2 -$$

$$5M^2 \beta^4 C^2 \sigma_4 + M^4 \sigma_5 + 10 M^4 \beta^2 C^2 \sigma_6 - \beta^4 \sigma_2 - 5\beta^6 C^2 A_3 -$$

$$M^2 \beta^2 \sigma_7 + 10 M^2 \beta^4 C^2 \sigma_8) + \frac{16 M^2 k y^4}{15 \beta^6 C^4} (2 M^2 \beta^2 \sigma_2 +$$

$$5M^2 \beta^4 C^2 \sigma_4 + 2 M^4 \sigma_5 + 5 M^4 \beta^2 C^2 \sigma_6 - 2\beta^4 \sigma_2 + 5\beta^6 C^2 A_3 -$$

$$2M^2 \beta^2 \sigma_7 + 5 M^2 \beta^4 C^2 \sigma_8) + \frac{64 M^6 k^3 x_1^6}{7\beta^{10}} (M^2 \beta^2 \sigma_9 -$$

$$6 M^4 \sigma_{12} - \beta^4 \sigma_{15} + 6 M^2 \beta^2 \sigma_{13}) - \frac{64 M^6 k^3 x_1^4 y^2}{35 \beta^{10} C^2} (M^2 \beta^2 \sigma_9 -$$

$$7 M^2 \beta^4 C^2 \sigma_{10} + M^4 \sigma_{12} + 28 M^4 \beta^2 C^2 \sigma_{13} - \beta^4 \sigma_{15} +$$

$$7\beta^6 C^2 \sigma_{16} - M^2 \beta^2 \sigma_{18} - 28 M^2 \beta^4 C^2 \sigma_{19}) -$$

$$\frac{64 M^6 k^3 x_1^2 y^4}{105 \beta^{10} C^4} (4 M^2 \beta^2 \sigma_9 + 7 M^2 \beta^4 C^2 \sigma_{10} - 35 M^2 \beta^6 C^4 \sigma_{11} +$$

$$4 M^4 \sigma_{12} + 7 M^4 \beta^2 C^2 \sigma_{13} + 70 M^4 \beta^4 C^4 \sigma_{14} - 4\beta^4 \sigma_{15} -$$

$$7\beta^6 C^2 \sigma_{16} + 35\beta^8 C^4 \sigma_{17} - 4 M^2 \beta^2 \sigma_{18} - 7 M^2 \beta^4 C^2 \sigma_{19} -$$

$$70 M^2 \beta^6 C^4 \sigma_{20}) - \frac{64 M^6 k^3 y^6}{105 \beta^{10} C^6} (8 M^2 \beta^2 \sigma_9 + 14 M^2 \beta^4 C^2 \sigma_{10} +$$

$$35 M^2 \beta^6 C^4 \sigma_{11} + 8 M^4 \sigma_{12} + 14 M^4 \beta^2 C^2 \sigma_{13} + 35 M^4 \beta^4 C^4 \sigma_{14} -$$

$$8\beta^4 \sigma_{15} - 14\beta^6 C^2 \sigma_{16} - 35\beta^8 C^4 \sigma_{17} - 8 M^2 \beta^2 \sigma_{18} -$$

$$\left. 14 M^2 \beta^4 C^2 \sigma_{19} - 35 M^2 \beta^6 C^4 \sigma_{20}) \right] - 2x_0 (M_2 + L_4 + 2 x_0 L_2)$$

$$(40d)$$

In equation (37), for example, the quantity $(L_1 + iL_2)$ is the lift-force coefficient associated with vertical motion of the wing. The real part L_1 is in phase with the vertical position of the wing and the complex part L_2 is 90° out of phase with this position. Similar definitions apply to the lift coefficient $(L_3 + iL_4)$ associated with pitching motion and to the moment coefficients $(M_1 + iM_2)$ and $(M_3 + iM_4)$. The imaginary or out-of-phase terms determine the aerodynamic damping associated with different wing motions.

Although the expressions for the components of lift-force and moment coefficients in equations (38) and (40), respectively, are lengthy, they may be quite easily evaluated with the aid of table I or the graphs in figures 3 and 4.
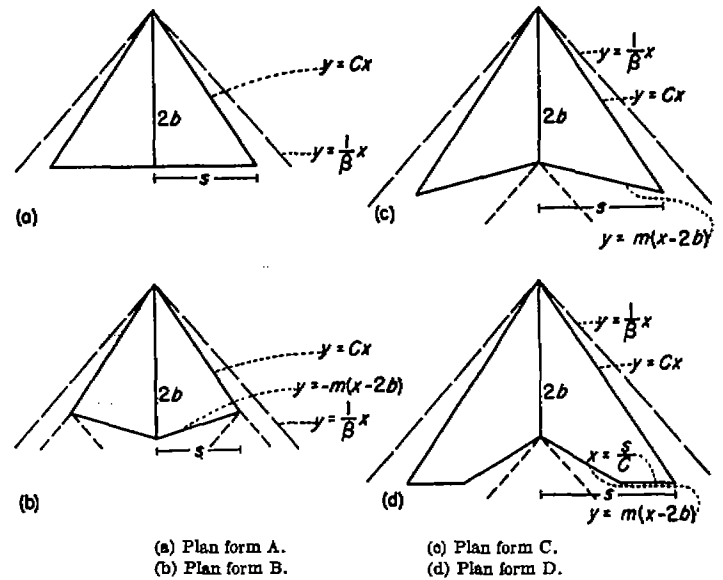

(a) Plan form A.   (c) Plan form C.
(b) Plan form B.   (d) Plan form D.

FIGURE 5.—Sketches illustrating different plan forms for which the force equations (38) and moment equations (40) apply.
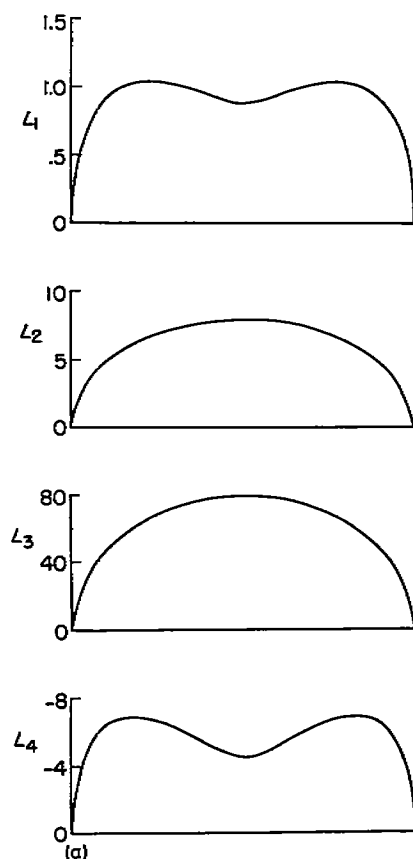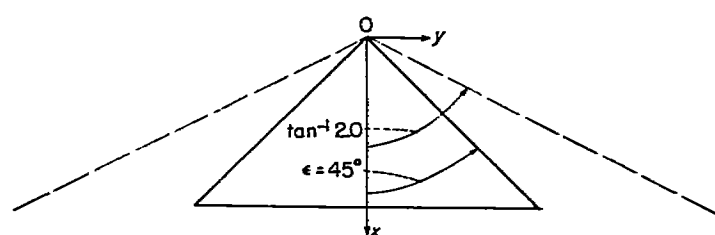
DISCUSSION

SAMPLE CALCULATIONS

In order to give some indication as to the general nature of the spanwise distribution of the different components of lift and moment coefficients, equations (38) and (40) have been evaluated at different spanwise positions y for plan form A (fig. 5(a)) for the following set of conditions: $\beta C = 0.5$, $C = 1.0$, $x_0 = 0.6$, $M = \sqrt{5/4}$, and $k = 0.1$. These sample results are plotted as functions of spanwise position in figure 6. The spanwise variations of the different components of lift force are shown in figure 6(a) and the corresponding variations of moment coefficients, in figure 6(b).

In figure 6 note that, for the particular set of conditions for this example, the maximum numerical values of the components of moment coefficients M_1, M_2, and M_3 are positive and are near the tips of the wing, whereas the maximum numerical value of the component M_4 is negative but is also near the tips. It may also be noted that the integrated (in spanwise direction) values of the components of moment coefficient, or components of total moment coefficient, would in each case have the same sign as the maximum value of the corresponding component of section moment coefficient. This result is not necessarily true in general, because changing some of the parameters involved in the evaluation of the spanwise distribution of some components of both force and moment coefficients may change the distributions significantly from those shown in figure 6.

The fact that the total component of moment coefficient M_4 is negative in the example just discussed shows that, for the conditions of the example, this term would not contribute

FIGURE 6.—Spanwise distribution of components of lift force and moment coefficients for $x_0=0.6$. $\beta C=0.5$; $C=1.0$; $k=0.1$; $M=\sqrt{5/4}$.

(a) Lift force.

to the aerodynamic damping but, on the contrary, would act as a source of energy for the oscillating system. This circumstance is significant since it leads to the possibility of the single-degree-of-freedom torsional instability discussed in the following section.

UNDAMPED TORSIONAL OSCILLATIONS

The wing plan forms discussed herein, like two-dimensional and rectangular wings, exhibit the possibility of undamped torsional oscillations for certain ranges of Mach number M and location of axis of rotation x_0. This fact is borne out, as indicated in the preceding paragraph, by considering the integrated (spanwise) value of the component of damping moment M_4 associated with pitching or torsional motions.
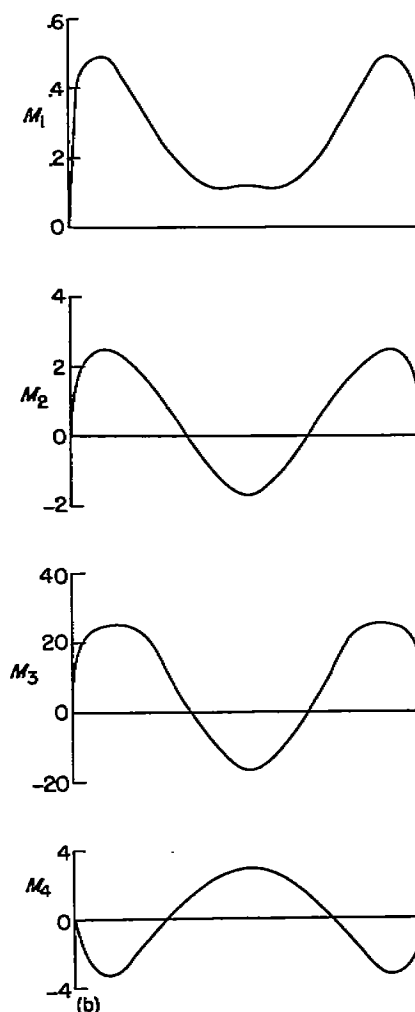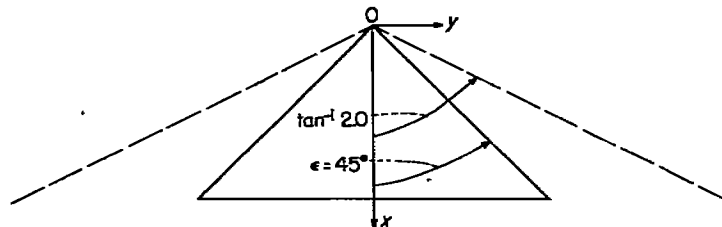
(b) Moment.

FIGURE 6.—Concluded.

The main results of this phenomenon can be obtained by considering very slow oscillations so that only terms in equation (40d) for M_4 involving the reduced frequency k to the order $1/k$ need be retained. In this case,

$$M_4=\frac{\sqrt{C^2 x_1{}^2-y^2}}{k}\left\{-\frac{4 x_1{}^2}{3\beta^2}[(M^2+1)A_0-A_1(4M^2-2)]-\right.$$

$$\left.\frac{4(2M^2-1)}{3\beta^2 C^2}\sigma_1 y^2+\frac{4 x_0}{\beta^2}[(A_0-(2M^2-1)A_1)x_1+\beta^2 x_0 A_0]\right\}$$

$$(41)$$

For the triangular plan form (fig. 5 (a)) the integrated value or total component of moment coefficient is

$$\overline{M}_4 = \frac{\pi C^2}{2\beta^2 k}\{-(2M^2+1)A_0 + 3(2M^2-1)A_1 +$$

$$4x_0[A_0-(2M^2-1)A_1]+4x_0^2(M^2-1)A_0\}\quad(42)$$

In general, the condition of torsional stability or instability depends on the sign and magnitude of $\overline{M}_4$ in equation (42). Positive values of $\overline{M}_4$ indicate stable conditions and negative values indicate the possibility of torsional instability. Between the stable and unstable conditions—that is, when $\overline{M}_4$ vanishes—a borderline state of unstable equilibrium separating damped and undamped torsional oscillations exists.

The ranges of values of Mach number M and location of axis of rotation x_0 for which $\overline{M}_4$ vanishes for some selected values of $C=\tan \epsilon$ are shown plotted in figure 7. The regions inside the curve in this figure indicate instability. The dashed curve, on which some of the solid curves terminate, represents the locus of values of M and x_0 for which

$\overline{M}_4$ vanishes for the whole class of triangular wings with supersonic leading edges; that is, for triangular wings where $\beta C \geq 1$ (see fig. 5 of reference 2). It will be noted that $\overline{M}_4$ vanishes for values of x_0 ahead of the root $\frac{3}{4}$-chord position. It will also be noted that, as the vertex angle $\epsilon = \tan^{-1} C$ decreases to 30°, the range of values of Mach number for which $\overline{M}_4$ vanishes decreases sharply.

In conclusion, investigation of equation (42) shows that, for a given value of the reduced frequency k, Mach number M, and location of the axis of rotation x_0, the magnitude of the damping coefficient $\overline{M}_4$ generally decreases as C decreases and, consequently, torsional instability is less likely to occur with slender triangles than with wider triangles.

LANGLEY AERONAUTICAL LABORATORY,
  NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
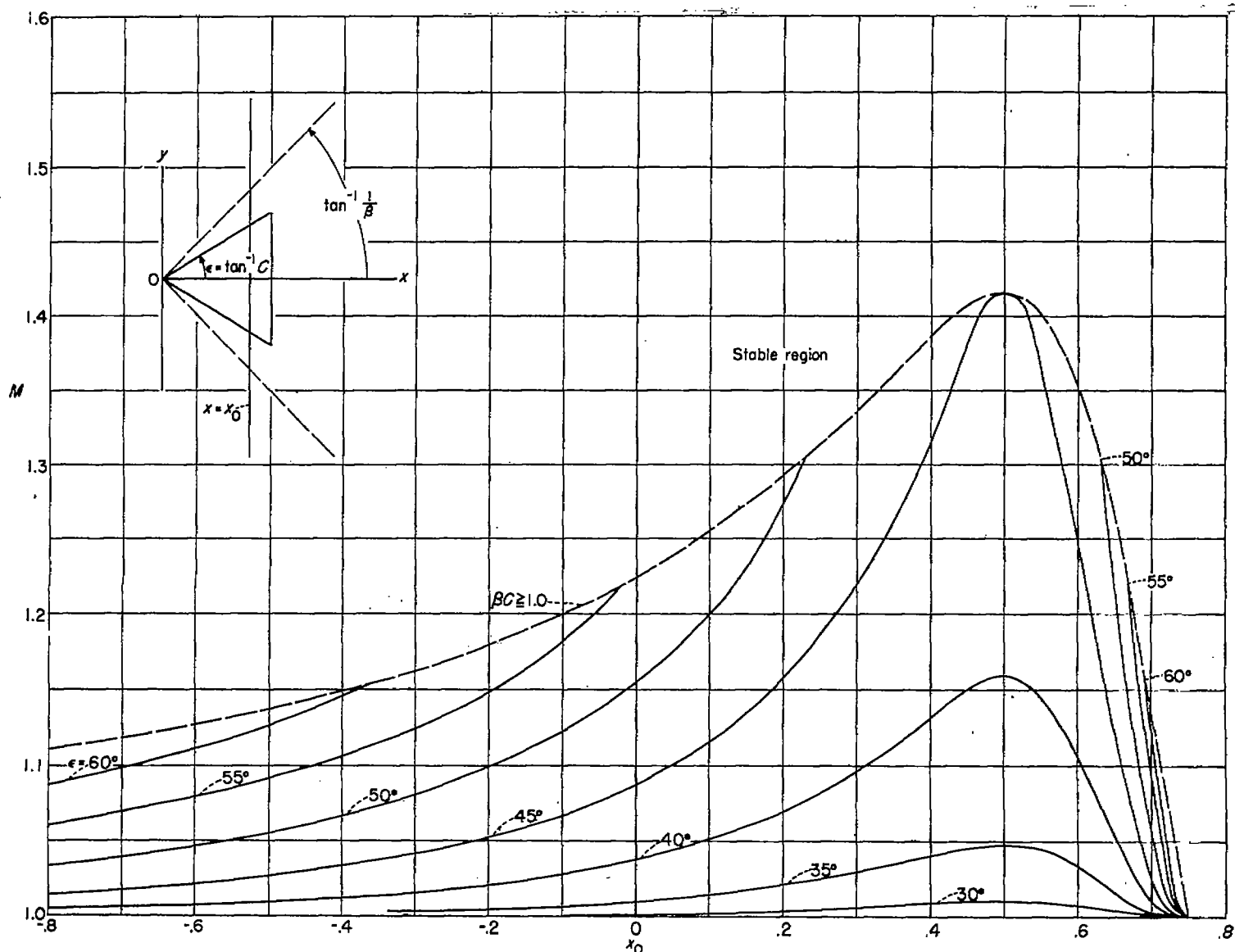 LANGLEY FIELD, VA., *June 19, 1951.*

FIGURE 7.—Curves showing ranges of Mach number M and axis of rotation x_0 for which the aerodynamic torsional damping moment vanishes for some selected values of angle ϵ.

APPENDIX

METHOD FOR DETERMINING DISTRIBUTION FUNCTIONS

In this appendix a method of obtaining the distribution functions is developed in detail. Expressions for the distribution functions required to drive the velocity potential to the fifth power of the frequency of oscillations and a set of functions useful in writing the expression for the potential in concise form is given.

As indicated in references 9 and 10, where the triangular wing is treated for constant angle of attack and for constant rolling and pitching motions, a convenient form of the velocity potential corresponding to a distribution of vertical velocity proportional to x^n is

$$\phi_n = \frac{1}{\pi}\frac{\partial}{\partial z}\int_{-c}^{c} F_n(\sigma)d\sigma \int_0^{\xi_1}\frac{\xi^{n+2}d\xi}{\sqrt{(x-\xi)^2-\beta^2(y-\sigma\xi)^2-\beta^2 z^2}} \tag{A1}$$

where σ is the slope of a ray passing through the vertex of the wing ($\eta=\sigma\xi$), ξ_1 is the least value of ξ that causes the denominator R in the integrand to vanish, and $F_n(\sigma)=\frac{1}{\xi}D_n(\xi,\eta)$ is the distribution function that is to be determined so that

$$(w_n)_{z=0}=\left(\frac{\partial\phi_n}{\partial z}\right)_{z=0}=x^n \tag{A2}$$

In equation (A1), the integrand is noted to be singular at the limit $\xi=\xi_1$. A form of the integrand which avoids this difficulty is obtained by making the following change of variables:

$$\left.\begin{array}{l}
q=\dfrac{x}{1-\beta^2\sigma^2}(1-\beta^2\sigma\theta)\\[2mm]
N=\dfrac{\beta x}{1-\beta^2\sigma^2}\sqrt{(\theta-\sigma)^2+\dfrac{z^2}{x^2}(1-\beta^2\sigma^2)}\\[2mm]
\theta=\dfrac{y}{x}\\[2mm]
\tau=\cosh^{-1}\dfrac{q-\xi}{N}
\end{array}\right\} \tag{A3}$$

where τ is the new variable of integration. With these substitutions equation (A1) becomes

$$\phi_n=\frac{1}{\pi}\frac{\partial}{\partial z}\int_{-c}^{c}\frac{F_n(\sigma)}{\sqrt{1-\beta^2\sigma^2}}\int_0^{\cosh^{-1}\frac{q}{N}}(q-N\cosh\tau)^{n+2}d\tau \tag{A4}$$

and the corresponding expression for $w_n=\left(\frac{\partial\phi_n}{\partial z}\right)_{z=0}$, which is an integral equation for $F_n(\sigma)$, is

$$w_n=\frac{1}{\pi}\lim_{z\to 0}\left\{-\beta^2(n+2)\int_{-c}^{c}\frac{F_n(\sigma)}{(1-\beta^2\sigma^2)^{3/2}}\int_0^{\cosh^{-1}\frac{q}{N}}N^n\left(\frac{q}{N}-\cosh\tau\right)^{n+1}\cosh\tau\,d\tau+\right.$$

$$\left.\beta^4 z^2(n+2)\int_{-c}^{c}\frac{F_n(\sigma)d\sigma}{(1-\beta^2\sigma^2)^{5/2}}\int_0^{\cosh^{-1}\frac{q}{N}}N^{n-2}\left(\frac{q}{N}-\cosh\tau\right)^n\left(\frac{q}{N}+n\cosh\tau\right)\cosh\tau\,d\tau\right\}$$

$$=x^n \tag{A5}$$

In this equation the value of w_n obtained by performing the indicated integrations and then going to the limit $z=0$ is the same as the value that would be obtained by first going to the limit and then performing the integration but neglecting singularities, pointed out subsequently, that arise when the value of σ approaches the value of θ. Making use of this fact reduces the

calculations involved and simplifies the integral equation for F_n. Thus passing to the limit gives for equation (A5)

$$w_n=-\frac{\beta^2(n+2)}{\pi}\int_{-C}^{C}\frac{F_n(\sigma)d\sigma}{(1-\beta^2\sigma^2)^{3/2}}\int_0^{\cosh^{-1}\frac{q}{N_0}}N_0{}^n\left(\frac{q}{N_0}-\cosh\,\tau\right)^{n+1}\cosh\,\tau\,d\tau$$
$$=x^n \tag{A6}$$

where N_0 is the value of N at $z=0$. The requirement that the normal velocity be proportional to x^n and independent of y implies that all derivatives of equation (A6) with respect to y or θ must vanish or that $F_n(\sigma)$ must be so determined that the final value of the integral in this equation be independent of θ. The requirement that all derivatives of equation (A6) with respect to θ vanish leads to other equations for F_n and, after $n+1$ such differentiations, the equations acquire forms for which solutions are known. The value of F_n can then be determined from these known solutions by evaluating each of the derivatives $\frac{d^K w_n}{d\theta^K}$ at any arbitrary value of θ in the range $-C<\theta<C$ as follows:

The Kth derivative of w_n with respect to θ (equation (A6)) gives the following integral equation for F_n:

$$\frac{d^K w_n}{d\theta^K}=-\frac{\beta^2(n+2)}{\pi}\int_{-C}^{C}\frac{F_n(\sigma)d\sigma}{(1-\beta^2\sigma^2)^{3/2}}\int_0^{\cosh^{-1}\frac{q}{N_0}}\left(\frac{\beta x}{1-\beta^2\sigma^2}\right)^K(q-N_0\cosh\,\tau)^{n+1}\frac{\cosh\,\tau}{N_0}\Phi(\tau)d\tau=0 \tag{A7}$$

where

$$\Phi(\tau)=\frac{(n+1)!}{(n+1-K)!}\left(\frac{\beta\sigma+\cosh\,\tau}{N_0\cosh\,\tau-q}\right)^K-\frac{K(n+1)!}{(n+2-K)!}\left(\frac{\beta\sigma+\cosh\,\tau}{N_0\cosh\,\tau-q}\right)^{K-1}N_0{}^{-1}+$$

$$\frac{K(K-1)(n+1)!}{(n+3-K)!}\left(\frac{\beta\sigma+\cosh\,\tau}{N_0\cosh\,\tau-q}\right)^{K-2}N_0{}^{-2}+\ldots+(-1)^K K!N_0{}^{-K}$$

When $K=n+1$, the expression for $\Phi(\tau)$ may be recognized as being a binomial expression, namely

$$\Phi(\tau)=(n+1)!\left(\frac{\beta\sigma+\cosh\,\tau}{N_0\cosh\,\tau-q}-\frac{1}{N_0}\right)^{n+1}$$
$$=\frac{(n+1)!(\beta\sigma N_0+q)^{n+1}}{(-N_0)^{n+1}(q-N_0\cosh\,\tau)^{n+1}}$$
$$=\frac{(n+1)!x^{n+1}}{(-N_0)^{n+1}(q-N_0\cosh\,\tau)^{n+1}} \tag{A8}$$

Thus the integral equation corresponding to the $n+1$ derivative of w_n is

$$\frac{d^{n+1}w_n}{d\theta^{n+1}}=\frac{(-1)^{n+1}(n+2)!\beta^{n+3}x^{2n+2}}{\pi}\int_{-C}^{C}\frac{F_n(\sigma)\,d\sigma}{N_0{}^{n+2}(1-\beta^2\sigma^2)^{n+\frac{5}{2}}}\int_0^{\sinh^{-1}\frac{x\sqrt{1-\beta^2\theta^2}}{N_0\sqrt{1-\beta^2\sigma^2}}}\cosh\,\tau\,d\tau$$
$$=\frac{(-1)^{n+1}(n+2)!x^n\sqrt{1-\beta^2\theta^2}}{\pi}\int_{-C}^{C}\frac{F_n(\sigma)}{(\theta-\sigma)^{n+3}}\,d\sigma=0 \tag{A9}$$

Further differentiation of w_n leads to other equations involving integrals similar to that in equation (A9) and, as will be seen subsequently, is not necessary for the determination of F_n. The singularity at $\sigma=\theta$ in equation (A9) is a result of going to the limit $z=0$ after equation (A5) and, as previously implied, is to be ignored.

Consider the following equation similar to equation (A9):

$$\int_{-C}^{C}\frac{f_n(\sigma)\,d\sigma}{(\theta-\sigma)^{n+3}}=0 \tag{A10}$$

It is known by analogy with problems in incompressible flow and may be shown by direct substitution and reduction that this equation is satisfied for any value of n $(n=0,1,2,\ldots)$ by the function

$$f_0(\sigma)=\sqrt{C^2-\sigma^2} \tag{A11}$$

This function also satisfies equation (A9) for any value of n and satisfies equation (A7) for all derivatives $\frac{d^K w_n}{d\theta^K}$ beyond the nth derivative. A more general form of solution to equation (A10) may be shown to be

$$f_m=\sigma^m\sqrt{C^2-\sigma^2} \tag{A12}$$

where m is an integer and $m \leq n$. The validity of this solution follows, since

$$\sigma^m = [\theta - (\theta - \sigma)]^m = \sum_{r=0}^{m} \frac{m! \, \theta^{m-r}}{(m-r)! \, r!} (\theta - \sigma)^r \tag{A13}$$

and

$$\frac{\sigma^m \sqrt{C^2 - \sigma^2}}{(\theta - \sigma)^{n+3}} = \sum_{r=0}^{m} \frac{m! \, \theta^{m-r}}{(m-r)! \, r!} \frac{\sqrt{C^2 - \sigma^2}}{(\theta - \sigma)^{n-r+3}} \tag{A14}$$

Each term of the summation in equation (A14) is found to have the form of the integrand in equation (A10), from which the function (A12) is concluded to satisfy equation (A10) and equation (A7) for all values of $K > n$.

From the foregoing discussion and consideration the distribution function $F_n(\sigma)$ may be uniquely determined in terms of expression (A12) as follows:

Consider the expression

$$F_n(\sigma) = B_0 f_0 + B_1 f_1 + \ldots + B_n f_n \tag{A15}$$

where the coefficients B_0, B_1, $\ldots$ B_n are constants that are to be determined.

Each term on the right of equation (A15) is noted to satisfy equation (A7) for all derivatives beyond the nth and a total of $n+1$ parameters is to be determined. If expression (A15) is introduced into equations (A6) and (A7) and the indicated integrations in equation (A6) and in equation (A7) for $K = 1, 2, \ldots n$ are performed, $n+1$ linear equations are obtained in B_0, B_1, $\ldots$ B_n from which these constants may be determined.

The integrations with respect to σ in equations (A6) and (A7) are in general difficult and tedious to perform; however, as previously pointed out it is only necessary to perform the integrations for some particular value of θ in the range $-C < \theta < C$. The integrals have their simplest form when $\theta = 0$ and the integrations can be made for this value of θ by reductions and use of formulas in reference 12.

The functions $\overline{D}_n$ and $\overline{\overline{D}}_n$ may also be determined by the method discussed for determining the functions D_n. The conditions to be satisfied by $\overline{D}_n$ and $\overline{\overline{D}}_n$, however, are determined by the value of w_2, equation (18), and its derivatives with regard to θ at $\theta = 0$.

The distribution functions D_n ($n = 0, 1, 2, 3, 4, 5$), $\overline{D}_n$ ($n = 0, 1, 2, 3$), and $\overline{\overline{D}}_n$ ($n = 0, 1$) calculated by the foregoing method are as follows:

$$D_0 = A_0 \sqrt{C^2 \xi^2 - \eta^2}$$

$$D_1 = A_1 \sqrt{C^2 \xi^2 - \eta^2}$$

$$D_2 = \left(A_2 + A_3 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

$$D_3 = \left(A_4 + A_5 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

$$D_4 = \left(A_6 + A_7 \frac{\beta^2 \eta^2}{\xi^2} + A_8 \frac{\beta^4 \eta^4}{\xi^4} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

$$D_5 = \left(A_9 + A_{10} \frac{\beta^2 \eta^2}{\xi^2} + A_{11} \frac{\beta^4 \eta^4}{\xi^4} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

$$\overline{D}_0 = \left(\overline{A}_0 + \overline{A}_1 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

$$\overline{D}_1 = \left(\overline{A}_2 + \overline{A}_3 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

$$\overline{D}_2 = \left(\overline{A}_4 + \overline{A}_5 \frac{\beta^2 \eta^2}{\xi^2} + \overline{A}_6 \frac{\beta^4 \eta^4}{\xi^4} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

$$\overline{D}_3 = \left(\overline{A}_7 + \overline{A}_8 \frac{\beta^2 \eta^2}{\xi^2} + \overline{A}_9 \frac{\beta^4 \eta^4}{\xi^4} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

$$\overline{\overline{D}}_0 = \left(\overline{\overline{A}}_0 + \overline{\overline{A}}_1 \frac{\beta^2 \eta^2}{\xi^2} + \overline{\overline{A}}_2 \frac{\beta^4 \eta^4}{\xi^4} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

$$\overline{\overline{D}}_1 = \left(\overline{\overline{A}}_3 + \overline{\overline{A}}_4 \frac{\beta^2 \eta^2}{\xi^2} + \overline{\overline{A}}_5 \frac{\beta^4 \eta^4}{\xi^4} \right) \sqrt{C^2 \xi^2 - \eta^2}$$

where, with the notations

$$W^2_{n,m} = -\frac{(n+2)\beta^2}{\pi x^n} \int_{-C}^{+C} \frac{\sigma^m \sqrt{C^2 - \sigma^2}}{(1 - \beta^2 \sigma^2)^{3/2}} d\sigma \lim_{\theta \to 0} \frac{\partial^p}{\partial \theta^p} \int_0^{\cosh^{-1} \frac{q}{N_0}} N_0^n \left(\frac{q}{N_0} - \cosh \tau \right)^{n+1} \cosh \tau \, d\tau$$

$$\overline{W}^2_{n,m} = \frac{\beta^2}{\pi x^{n+2}} \int_{-C}^{+C} \frac{\sigma^m \sqrt{C^2 - \sigma^2}}{\sqrt{1 - \beta^2 \sigma^2}} d\sigma \lim_{\theta \to 0} \frac{\partial^p}{\partial \theta^p} \int_0^{\cosh^{-1} \frac{q}{N_0}} (q - N_0 \cosh \tau)^{n+1} [2q - (n+4)N_0 \cosh \tau] \sinh^2 \tau \, d\tau$$

and

$$\overline{\overline{W}}^2_{n,m} = \frac{\beta^2}{\pi x^{n+4}} \int_{-C}^{+C} \sigma^m \sqrt{C^2 - \sigma^2} \sqrt{1 - \beta^2 \sigma^2} \, d\sigma \lim_{\theta \to 0} \frac{\partial^p}{\partial \theta^p} \int_0^{\cosh^{-1} \frac{q}{N_0}} N_0^2 (q - N_0 \cosh \tau)^{n+1} [4q - (n+6)N_0 \cosh \tau] \sinh^4 \tau \, d\tau$$

(Note that the above expressions will vanish identically unless p and m are either both odd or both even.)

the A's are given by the following equations:

$$A_0=\frac{1}{W^0_{0,0}}$$

$$A_1=\frac{1}{W^0_{1,0}}$$

$$\left.\begin{array}{l}A_2W^0_{2,0}+A_3W^0_{2,2}=1\\A_2W^2_{2,0}+A_3W^2_{2,2}=0\end{array}\right\}$$

$$\left.\begin{array}{l}A_4W^0_{3,0}+A_5W^0_{3,2}=1\\A_4W^2_{3,0}+A_5W^2_{3,2}=0\end{array}\right\}$$

$$\left.\begin{array}{l}A_6W^0_{4,0}+A_7W^0_{4,2}+A_8W^0_{4,4}=1\\A_6W^2_{4,0}+A_7W^2_{4,2}+A_8W^2_{4,4}=0\\A_6W^4_{4,0}+A_7W^4_{4,2}+A_8W^4_{4,4}=0\end{array}\right\}$$

$$\left.\begin{array}{l}A_9W^0_{5,0}+A_{10}W^0_{5,2}+A_{11}W^0_{5,4}=1\\A_9W^2_{5,0}+A_{10}W^2_{5,2}+A_{11}W^2_{5,4}=0\\A_9W^4_{5,0}+A_{10}W^4_{5,2}+A_{11}W^4_{5,4}=0\end{array}\right\}$$

$$\left.\begin{array}{l}\overline{A}_0W^0_{2,0}+\overline{A}_1W^0_{2,2}=A_0\overline{W}^0_{0,0}\\\overline{A}_0W^2_{2,0}+\overline{A}_1W^2_{2,2}=A_0\overline{W}^2_{0,0}\end{array}\right\}$$

$$\left.\begin{array}{l}\overline{A}_2W^0_{3,0}+\overline{A}_3W^0_{3,2}=A_1\overline{W}^0_{1,0}\\\overline{A}_2W^2_{3,0}+\overline{A}_3W^2_{3,2}=A_1\overline{W}^2_{1,0}\end{array}\right\}$$

$$\left.\begin{array}{l}\overline{A}_4W^0_{4,0}+\overline{A}_5W^0_{4,2}+\overline{A}_6W^0_{4,4}=A_2\overline{W}^0_{2,0}+A_3\overline{W}^0_{2,2}\\\overline{A}_4W^2_{4,0}+\overline{A}_5W^2_{4,2}+\overline{A}_6W^2_{4,4}=A_2\overline{W}^2_{2,0}+A_3\overline{W}^2_{2,2}\\\overline{A}_4W^4_{4,0}+\overline{A}_5W^4_{4,2}+\overline{A}_6W^4_{4,4}=A_2\overline{W}^4_{2,0}+A_3\overline{W}^4_{2,2}\end{array}\right\}$$

$$\left.\begin{array}{l}\overline{A}_7W^0_{5,0}+\overline{A}_8W^0_{5,2}+\overline{A}_9W^0_{5,4}=A_4\overline{W}^0_{3,0}+A_5\overline{W}^0_{3,2}\\\overline{A}_7W^2_{5,0}+\overline{A}_8W^2_{5,2}+\overline{A}_9W^2_{5,4}=A_4\overline{W}^2_{3,0}+A_5\overline{W}^2_{3,2}\\\overline{A}_7W^4_{5,0}+\overline{A}_8W^4_{5,2}+\overline{A}_9W^4_{5,4}=A_4\overline{W}^4_{3,0}+A_5\overline{W}^4_{3,2}\end{array}\right\}$$

$$\left.\begin{array}{l}\overline{\overline{A}}_0W^0_{4,0}+\overline{\overline{A}}_1W^0_{4,2}+\overline{\overline{A}}_2W^0_{4,4}=-\frac{A_0}{6}\overline{\overline{W}}^0_{0,0}+\overline{A}_0\overline{W}^0_{2,0}+\overline{A}_1\overline{W}^0_{2,2}\\\overline{\overline{A}}_0W^2_{4,0}+\overline{\overline{A}}_1W^2_{4,2}+\overline{\overline{A}}_2W^2_{4,4}=-\frac{A_0}{6}\overline{\overline{W}}^2_{0,0}+\overline{A}_0\overline{W}^2_{2,0}+\overline{A}_1\overline{W}^2_{2,2}\\\overline{\overline{A}}_0W^4_{4,0}+\overline{\overline{A}}_1W^4_{4,2}+\overline{\overline{A}}_2W^4_{4,4}=-\frac{A_0}{6}\overline{\overline{W}}^4_{0,0}+\overline{A}_0\overline{W}^4_{2,0}+\overline{A}_1\overline{W}^4_{2,2}\end{array}\right\}$$

$$\left.\begin{array}{l}\overline{\overline{A}}_3W^0_{5,0}+\overline{\overline{A}}_4W^0_{5,2}+\overline{\overline{A}}_5W^0_{5,4}=-\frac{A_1}{6}\overline{\overline{W}}^0_{1,0}+\overline{A}_2\overline{W}^0_{3,0}+\overline{A}_3\overline{W}^0_{3,2}\\\overline{\overline{A}}_3W^2_{5,0}+\overline{\overline{A}}_4W^2_{5,2}+\overline{\overline{A}}_5W^2_{5,4}=-\frac{A_1}{6}\overline{\overline{W}}^2_{1,0}+\overline{A}_2\overline{W}^2_{3,0}+\overline{A}_3\overline{W}^2_{3,2}\\\overline{\overline{A}}_3W^4_{5,0}+\overline{\overline{A}}_4W^4_{5,2}+\overline{\overline{A}}_5W^4_{5,4}=-\frac{A_1}{6}\overline{\overline{W}}^4_{1,0}+\overline{A}_2\overline{W}^4_{3,0}+\overline{A}_3\overline{W}^4_{3,2}\end{array}\right\}$$

where, with $\beta C=\rho_0$,

$$W^0_{0,0}=E'$$

$$W^0_{1,0}=\frac{\rho_0^2F'+(1-2\rho_0^2)E'}{(1-\rho_0^2)}$$

$$W^0_{2,0}=\frac{(5\rho_0^2-3\rho_0^4)F'+(2-10\rho_0^2+6\rho_0^4)E'}{2(1-\rho_0^2)^2}$$

$$W^0_{2,2}=\frac{2\rho_0^4F'-(\rho_0^2+\rho_0^4)E'}{2\beta^2(1-\rho_0^2)^2}$$

$$W^2_{2,0}=\frac{2\beta^2}{C^2}W^0_{2,2}$$

$$W^2_{2,2}=\frac{(5\rho_0^4-3\rho_0^2)F'+(6-10\rho_0^2+2\rho_0^4)E'}{(1-\rho_0^2)^2}$$

$$W^0_{3,0}=\frac{(27\rho_0^2-31\rho_0^4+12\rho_0^6)F'+(6-55\rho_0^2+65\rho_0^4-24\rho_0^6)E'}{6(1-\rho_0^2)^3}$$

$$W^0_{3,2}=\frac{(9\rho_0^4-\rho_0^6)F'-(3\rho_0^2+7\rho_0^4-2\rho_0^6)E'}{6\beta^2(1-\rho_0^2)^3}$$

$$W^2_{3,0}=\frac{6\beta^2}{C^2}W^0_{3,2}$$

$$W^2_{3,2}=\frac{(2\rho_0^6+9\rho_0^4-3\rho_0^2)F'+(6-15\rho_0^2+5\rho_0^4-4\rho_0^6)E'}{(1-\rho_0^2)^3}$$

$$W^0_{4,0}=\frac{(56\rho_0^2-92\rho_0^4+72\rho_0^6-20\rho_0^8)F'+(8-117\rho_0^2+202\rho_0^4-149\rho_0^6+40\rho_0^8)E'}{8(1-\rho_0^2)^4}$$

$$W^0_{4,2}=\frac{(17\rho_0^4-2\rho_0^6+\rho_0^8)F'-(4\rho_0^2+18\rho_0^4-8\rho_0^6+2\rho_0^8)E'}{8\beta^2(1-\rho_0^2)^4}$$

$$W^0_{4,4}=\frac{(8\rho_0^6+8\rho_0^8)F'-(\rho_0^4+14\rho_0^6+\rho_0^8)E'}{8\beta^4(1-\rho_0^2)^4}$$

$$W^2_{4,0}=\frac{12\beta^2}{C^2}W^0_{4,2}$$

$$W^2_{4,2}=-\frac{(6\rho_0^2-30\rho_0^4-30\rho_0^6+6\rho_0^8)F'-(12-39\rho_0^2+6\rho_0^4-39\rho_0^6+12\rho_0^8)E'}{2(1-\rho_0^2)^4}$$

$$W^2_{4,4}=\frac{(3\rho_0^4-6\rho_0^6+51\rho_0^8)F'-(6\rho_0^2-24\rho_0^4+54\rho_0^6+12\rho_0^8)E'}{2\beta^2(1-\rho_0^2)^4}$$

$$W^4_{4,0}=\frac{24\beta^4}{C^4}W^0_{4,4}$$

$$W^4_{4,2}=\frac{2\beta^2}{C^2}W^2_{4,4}$$

$$W^4_{4,4}=\frac{(168\rho_0^8-276\rho_0^6+216\rho_0^4-60\rho_0^2)F'+(24\rho_0^8-351\rho_0^6+606\rho_0^4-447\rho_0^2+120)E'}{(1-\rho_0^2)^4}$$

$$W^0_{5,0}=\frac{(400\rho_0^2-833\rho_0^4+994\rho_0^6-553\rho_0^8+120\rho_0^{10})F'+(40-859\rho_0^2+1910\rho_0^4-2115\rho_0^6+1136\rho_0^8-240\rho_0^{10})E'}{40(1-\rho_0^2)^5}$$

$$W^0_{5,2}=\frac{(115\rho_0^4-2\rho_0^6+19\rho_0^8-4\rho_0^{10})F'-(20\rho_0^2+151\rho_0^4-74\rho_0^6+39\rho_0^8-8\rho_0^{10})E'}{40\beta^2(1-\rho_0^2)^5}$$

$$W^0_{5,4}=\frac{(55\rho_0^6+74\rho_0^8-\rho_0^{10})F'-(5\rho_0^4+108\rho_0^6+17\rho_0^8-2\rho_0^{10})E'}{40\beta^4(1-\rho_0^2)^5}$$

$$W^2_{5,0}=\frac{20\beta^2}{C^2}W^0_{5,2}$$

$$W^2_{5,2}=\frac{(8\rho_0^{10}-35\rho_0^8+112\rho_0^6+49\rho_0^4-6\rho_0^2)F'+(12-47\rho_0^2-22\rho_0^4-127\rho_0^6+72\rho_0^8-16\rho_0^{10})E'}{2(1-\rho_0^2)^5}$$

$$W^2_{5,4}=\frac{(3\rho_0^4-2\rho_0^6+131\rho_0^8-4\rho_0^{10})F'-(6\rho_0^2-31\rho_0^4+108\rho_0^6+53\rho_0^8-8\rho_0^{10})E'}{2\beta^2(1-\rho_0^2)^5}$$

$$W^4_{5,0}=\frac{120\beta^4}{C^4}W^0_{5,4}$$

$$W^4_{5,2}=\frac{6\beta^2}{C^2}W^2_{5,4}$$

$$W^4_{5,4}=\frac{(24\rho_0^{10}+609\rho_0^8-462\rho_0^6+273\rho_0^4-60\rho_0^2)F'+(120-561\rho_0^2+1026\rho_0^4-969\rho_0^6+48\rho_0^8-48\rho_0^{10})E'}{(1-\rho_0^2)^5}$$

$$\overline{W}^0_{0,0}=-\frac{\rho_0^2F'-\rho_0^2E'}{2(1-\rho_0^2)}$$

$$\overline{W}^2_{0,0}=-\frac{\beta^2(\rho_0^2F'-E')}{(1-\rho_0^2)}$$

$$\overline{W}^0_{1,0}=-\frac{(3\rho_0^2-\rho_0^4)F'-(4\rho_0^2-2\rho_0^4)E'}{6(1-\rho_0^2)^2}$$

$$\overline{W}^2_{1,0}=-\frac{2\beta^2}{C^2}W^0_{2,2}$$

$$\overline{W}^0_{2,0}=-\frac{(12\rho_0^2-7\rho_0^4+3\rho_0^6)F'-(19\rho_0^2-17\rho_0^4+6\rho_0^6)E'}{24(1-\rho_0^2)^3}$$

$$\overline{W}^0_{2,2}=-\frac{(3\rho_0^4+5\rho_0^6)F'-(7\rho_0^4+\rho_0^6)E'}{24\beta^2(1-\rho_0^2)^3}$$

$$\overline{W}^2_{2,0}=-\frac{\beta^2[(7\rho_0^2+\rho_0^4)F'-(2+7\rho_0^2-\rho_0^4)E']}{2(1-\rho_0^2)^3}$$

$$\overline{W}^2_{2,2}=-\frac{(\rho_0^4+7\rho_0^6)F'+(\rho_0^2-7\rho_0^4-2\rho_0^6)E'}{2(1-\rho_0^2)^3}$$

$$\overline{W}^4_{2,0}=-\frac{\beta^4[(5\rho_0^2+3\rho_0^4)F'-(1+7\rho_0^4)E']}{(1-\rho_0^2)^3}$$

$$\overline{W}'_{2,2} = -\frac{\beta^2[(3\rho_0{}^2-7\rho_0{}^4+12\rho_0{}^6)F'-(6-17\rho_0{}^2+19\rho_0{}^4)E']}{(1-\rho_0{}^2)^3}$$

$$\overline{W}'_{3,0} = -\frac{(60\rho_0{}^2-44\rho_0{}^4+44\rho_0{}^6-12\rho_0{}^8)F'-(107\rho_0{}^2-126\rho_0{}^4+91\rho_0{}^6-24\rho_0{}^8)E'}{120(1-\rho_0{}^2)^4}$$

$$\overline{W}'_{3,2} = -\frac{(15\rho_0{}^4+34\rho_0{}^6-\rho_0{}^8)F'-(38\rho_0{}^4+12\rho_0{}^6-2\rho_0{}^8)E'}{120\beta^2(1-\rho_0{}^2)^4}$$

$$\overline{W}'^2_{3,0} = -\frac{\beta^2[(33\rho_0{}^2+14\rho_0{}^4+\rho_0{}^6)F'-(6+46\rho_0{}^2-6\rho_0{}^4+2\rho_0{}^6)E']}{6(1-\rho_0{}^2)^4}$$

$$\overline{W}'^2_{3,2} = -\frac{(6\rho_0{}^4+44\rho_0{}^6-2\rho_0{}^8)F'+(3\rho_0{}^2-34\rho_0{}^4-21\rho_0{}^6+4\rho_0{}^8)E'}{6(1-\rho_0{}^2)^4}$$

$$\overline{W}'_{3,0} = -W'_{4,0}$$

$$\overline{W}'_{3,2} = -W'_{4,2}$$

$$\overline{\overline{W}}'^0_{0,0} = \frac{\beta^2}{4}W'^0_{2,2}$$

$$\overline{\overline{W}}'^2_{0,0} = -\frac{3\beta^2[(\rho_0{}^2+\rho_0{}^4)F'-2\rho_0{}^2E']}{2(1-\rho_0{}^2)^2}$$

$$\overline{\overline{W}}'^4_{0,0} = \frac{6\beta^4}{C^2}W'^0_{2,2}$$

$$\overline{\overline{W}}'^0_{1,0} = \frac{3\beta^2}{20}W'^0_{3,2}$$

$$\overline{\overline{W}}'^2_{1,0} = -\frac{12\beta^2}{C^2}\overline{W}'^0_2$$

$$\overline{\overline{W}}'^4_{1,0} = -3\overline{W}'^4_{2,0}$$

Combinations of these functions useful in writing the potential in concise form are as follows:

$$\sigma_j = P_j + \frac{Q_j}{M^2} + \frac{R_j}{M^4}$$

where

$P_1 = A_0 - A_1$	$Q_1 = 0$	$R_1 = 0$
$P_2 = A_1 - A_2$	$Q_2 = 0$	$R_2 = 0$
$P_3 = \frac{1}{2}(A_0 - 2A_1 + A_2)$	$Q_3 = -\frac{\overline{A}_0}{2}$	$R_3 = 0$
$P_4 = \frac{A_3}{2}$	$Q_4 = -\frac{\overline{A}_1}{2}$	$R_4 = 0$
$P_5 = \frac{1}{6}(A_0 - 3A_1 + 3A_2 - A_4)$	$Q_5 = -\frac{1}{2}(\overline{A}_0 - \overline{A}_2)$	$R_5 = 0$
$P_6 = \frac{1}{6}(3A_3 - A_5)$	$Q_6 = -\frac{1}{2}(\overline{A}_1 - \overline{A}_3)$	$R_6 = 0$
$P_7 = \frac{1}{2}(A_1 - 2A_2 + A_4)$	$Q_7 = -\frac{\overline{A}_2}{2}$	$R_7 = 0$
$P_8 = \frac{1}{2}(2A_3 - A_5)$	$Q_8 = \frac{\overline{A}_3}{2}$	$R_8 = 0$

$$P_9 = \frac{1}{24}(A_0 - 4A_1 + 6A_2 - 4A_4 + A_8)$$

$$Q_9 = -\frac{1}{4}(\overline{A}_0 - 2\overline{A}_2 + \overline{A}_4)$$

$$R_9 = \frac{\overline{\overline{A}}_0}{4}$$

$$P_{10} = \frac{1}{24}(6A_3 - 4A_5 + A_7)$$

$$Q_{10} = -\frac{1}{4}(\overline{A}_1 - 2\overline{A}_3 + \overline{A}_5)$$

$$R_{10} = \frac{\overline{\overline{A}}_1}{4}$$

$$P_{11} = \frac{A_8}{24}$$

$$Q_{11} = -\frac{\overline{A}_6}{4}$$

$$R_{11} = \frac{\overline{\overline{A}}_2}{4}$$

$$P_{12} = \frac{1}{120}(A_0 - 5A_1 + 10A_2 - 10A_4 + 5A_6 - A_9)$$

$$Q_{12} = -\frac{1}{12}(\overline{A}_0 - 3\overline{A}_2 + 3\overline{A}_4 - \overline{A}_7)$$

$$R_{12} = \frac{1}{4}(\overline{\overline{A}}_0 - \overline{\overline{A}}_3)$$

$$P_{13} = \frac{1}{120}(10A_3 - 10A_5 + 5A_7 - A_{10})$$

$$Q_{13} = -\frac{1}{12}(\overline{A}_1 - 3\overline{A}_3 + 3\overline{A}_5 - \overline{A}_8)$$

$$R_{13} = \frac{1}{4}(\overline{\overline{A}}_1 - \overline{\overline{A}}_4)$$

$$P_{14} = \frac{1}{120}(5A_8 - A_{11})$$

$$Q_{14} = -\frac{1}{12}(3\overline{A}_6 - \overline{A}_9)$$

$$R_{14} = \frac{1}{4}(\overline{\overline{A}}_2 - \overline{\overline{A}}_5)$$

$$P_{15} = \frac{1}{6}(A_1 - 3A_2 + 3A_4 - A_6)$$

$$Q_{15} = -\frac{1}{2}(\overline{A}_2 - \overline{A}_4)$$

$$R_{15} = 0$$

$$P_{16} = \frac{1}{6}(3A_5 - 3A_3 - A_7)$$

$$Q_{16} = -\frac{1}{2}(\overline{A}_3 - \overline{A}_5)$$

$$R_{16} = 0$$

$$P_{17} = -\frac{A_8}{6}$$

$$Q_{17} = \frac{\overline{A}_6}{2}$$

$$R_{17} = 0$$

$$P_{18} = \frac{1}{24}(A_1 - 4A_2 + 6A_4 - 4A_6 + A_9)$$

$$Q_{18} = -\frac{1}{4}(\overline{A}_2 - 2\overline{A}_4 + \overline{A}_7)$$

$$R_{18} = \frac{\overline{\overline{A}}_3}{4}$$

$$P_{19} = \frac{1}{24}(A_{10} - 4A_7 + 6A_5 - 4A_3)$$

$$Q_{19} = -\frac{1}{4}(\overline{A}_3 - 2\overline{A}_5 + \overline{A}_8)$$

$$R_{19} = \frac{\overline{\overline{A}}_4}{4}$$

$$P_{20} = \frac{1}{24}(A_{11} - 4A_8)$$

$$Q_{20} = \frac{1}{4}(2\overline{A}_6 - \overline{A}_9)$$

$$R_{20} = \frac{\overline{\overline{A}}_5}{4}$$

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TABLE I.—VALUES OF A_i, P_i, Q_i, AND R_i FOR PARTICULAR VALUES OF $\rho_0 = \beta C$

(a) A_i

ρ_0	A_0	A_1	A_2	A_3
0	1.0000	1.0000		0.1667
.03408	.9974	.9920		.1638
.08710	.9875	.9600		.1582
.17365	.9615	.9015		.1511
.25853	.9290	.8348		.1100
.34202	.8941	.7680		.0928
.42262	.8603	.7078		.0781
.50000	.8287	.6562		.0650
.57358	.7945	.6101		.0550
.64279	.7600	.6717		.0510
.70711	.7404	.5304		.0465
.76604	.7178	.5122		.0411
.81915	.6982	.4807		.0370
.86603	.6814	.4712		.0348
.90631	.6575	.4503		.0327
.93969	.0476	.4355		.0208
1.00000	.0360	.4244		.0283

(b) P_i

(c) Q_j

ρ_s	Q_1	Q_2	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}	Q_{13}	Q_{14}	Q_{15}	Q_{16}	Q_{17}	Q_{18}	Q_{19}	Q_{20}

(d) R_j

ρ_s	R_9	R_{10}	R_{11}	R_{13}	R_{15}	R_{14}	R_{16}	R_{19}	R_{20}